NUWC-NL Technical Report 10,041
23 April 1992

Exact Detection Performance of Multiple-Pulse Frequency-Shift Signals in a Partially-Correlated Fading Medium with Generalized Noncentral Chi-Squared Statistics

Albert H. Nuttall
Surface ASW Directorate

Naval Undersea Warfare Center Detachment
New London, Connecticut

PREFACE

This research was conducted under NUWC Project Number A70272, Subproject Number RR0000-N01, **Selected Statistical Problems in Acoustic Signal Processing**, Principal Investigator Dr. Albert H. Nuttall (Code 302). This technical report was prepared with funds provided by the NUWC In-House Independent Research Program, sponsored by the Office of Naval Research. Also, this report was prepared under NUWC Project Number M20027, **Multistatic Sonar System**, Chief Engineer Ira Cohen (Code 3332), Program Manager Carl Mason (Code 304), NAVSEA Program Manager Darrel Spiers (Code 06UR1).

The technical reviewer for this report was Weita Chang (Code 3314).

REVIEWED AND APPROVED: 23 APRIL 1992

D. M. Mayer for

DAVID DENCE
Director, Surface Antisubmarine Warfare

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE 23 April 1992

3. REPORT TYPE AND DATES COVERED Progress

4. TITLE AND SUBTITLE Exact Detection Performance of Multiple-Pulse Frequency-Shift Signals in a Partially-Correlated Fading Medium with Generalized Noncentral Chi-Squared Statistics

5. FUNDING NUMBERS PE 61152N

6. AUTHOR(S)

Albert H. Nuttall

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Naval Undersea Warfare Center Detachment
New London, Connecticut 06320

8. PERFORMING ORGANIZATION REPORT NUMBER TR 10041

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Chief of Naval Research
Office of the Chief of Naval Research
Arlington, VA 22217-5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION / AVAILABILITY STATEMENT

Approved for public release;
distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

The transmitted signal consists of K pulses separated in time,frequency space so as to be nonoverlapping. In passing through the medium to the receiver, each signal pulse is subjected to fading. In particular, pulse pairs which are closely spaced in time,frequency space can fade in a highly dependent fashion, while those more widely separated can have relatively independent fading behavior; that is, the transmitted frequency-shift-keyed signal pulses undergo partially-correlated fading of a very general character that contains both deterministic components as well as random components. The amplitude-fading statistics are not limited to be Rayleigh.

Additive zero-mean Gaussian noise, which is stationary over the total signal transmission time and which has a flat spectrum over the total signal bandwidth, is present at the input to the receiver, in addition to the fluctuating signal pulses (when

14. SUBJECT TERMS

detection — performance
multiple pulses — partial correlation
FSK signals — fading medium

15. NUMBER OF PAGES 152

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

13. ABSTRACT (continued)

present).

Receiver processing consists of matched filtering of each of the K time-delayed, frequency-shifted pulse locations, followed by squared-envelope detection, sampling, summation, and comparison of this decision variable with a fixed threshold for a statement on signal presence or absence.

The characteristic function of the decision variable is derived in closed form for a very general model of partially-correlated fading that subsumes Rayleigh, Rician, and non-central Chi probability densities for the amplitude variate as special cases. This characteristic function depends on the number K of signal pulses, the number M of fading components, the deterministic received signal energy in each fading component, the average random received signal energy in each fading component, as well as the received noise spectral density level. Numerous special cases are pointed out and specific results are given in detail.

An efficient expansion for the exceedance distribution, for one of the cases, is listed and exercised for a representative numerical example. Comparisons with earlier approximations reveal them to have been pessimistic by several dB in ranges considered typical for practical applications. Also, the effects of correlated fading of the signal pulses are found to be not overly detrimental until the normalized covariance coefficient of adjacent pulses gets larger than approximately .5.

14. SUBJECT TERMS (continued)

chi-squared
random fading
matched filtering
characteristic function
deterministic
Gaussian noise
envelope detection
series expansions

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

FSK	frequency shift keying
t	time, (1)
$s(t)$	transmitted real signal waveform, (1)
$\underline{s}(t)$	transmitted signal complex envelope, (1)
f_o	carrier frequency, (1)
Re	real part, (1)
$\tilde{E}$	transmitted signal energy, (2)
r	amplitude scaling in transmission, (3)
θ	phase shift in transmission, (3)
t_d	time delay of received signal, (3)
f_d	frequency shift of received signal, (3)
overbar	ensemble average, (4)
$n(t)$	received noise waveform, (5)
$\underline{n}(t)$	received noise complex envelope, (5)
N_o	received noise one-sided spectral level, (5), (10)
A	local reference amplitude scaling, (6)
ϕ	local reference phase shift, (6)
$\alpha(t)$	complex envelope of received waveform, (7)
$\beta(t)$	complex envelope of local reference, (7)
γ	output of receiver processor, (8), figure 1
μ	parameter, $2\ r\ A\ \tilde{E}$, (8), (17)
n_r, n_i	real and imaginary noise components, (9)
$\delta(t)$	delta function, (10)
σ^2	parameter, $2\ N_o\ A^2\ \tilde{E}$, (12), (17)
Prob	exceedance probability, (14)

$f_c(\xi)$	conditional characteristic function of γ, (16)
$\chi(j)$	j-th conditional cumulant of γ, (19)
d	conditional deflection criterion, (21)
K	number of transmitted signal pulses, (22)
r_k	amplitude scaling of k-th signal pulse, (22)
θ_k	phase shift of k-th signal pulse, (22)
A_k	local reference scaling for k-th pulse, (22)
ϕ_k	local reference phase shift for k-th pulse, (22)
$\alpha_k(t)$	complex envelope of k-th received waveform, (22)
$\beta_k(t)$	complex envelope of k-th local reference, (22)
$\underline{s}_k(t)$	transmitted k-th signal pulse complex envelope, (22)
$\tilde{E}_k$	transmitted signal energy in k-th pulse, (23)
μ_k	k-th parameter, $2\ r_k\ A_k\ \tilde{E}_k$, (24)
σ_k^2	k-th parameter, $2\ N_o\ A_k^2\ \tilde{E}_k$, (24)
$f_c(\xi)$	conditional characteristic function of γ, (26), (27)
$\chi(j)$	j-th conditional cumulant of γ, (28)
d_K	conditional deflection of γ, (30)
$\overline{d_K}$	average deflection of output γ in figure 1, (32), (33)
A	common receiver weighting for K pulses, (35), (38)
$\tilde{E}$	common transmitted energy of K signal pulses, (36), (38)
S	sum of K power fading variates $\{r_k^2\}$, (37)
$f_S(\xi)$	characteristic function of random variable S, (41)
$f_\gamma(\xi)$	unconditional characteristic function of output γ, (42)
q_k	r_k^2, power scaling of k-th signal pulse, (43), (44)
t_k	time location of k-th FSK signal pulse, (44)
f_k	frequency location of k-th FSK signal pulse, (44)
q(t,f)	continuous fading process, time t and frequency f, (45)

M	number of fading components, (45)
$c_m(t,f)$	deterministic component of fading model, (45)
$g_m(t,f)$	random component of fading model, (45)
R_{mn}	covariance of g_m with g_n, (47)
τ,ν	delay and shift parameters in covariance R_{mn}, (47)
D_{km}	deterministic received signal energy in m-th component of k-th pulse, (50)
D_k	deterministic received signal energy in k-th pulse, (51)
E_{km}	average random received signal energy in m-th component of k-th pulse, (52)
E_k	average random received signal energy in k-th pulse,(53)
$\tilde{q}_k$	zero-mean component of q_k, (54)
$\tilde{R}_{kj}$	covariance between q_k and q_j, (55)
$\tilde{R}_{kk}$	variance of q_k, (56), (58)
ρ_{kj}	covariance coefficient between q_k and q_j, (56)
δ_{mn}	1 for m = n; 0 otherwise, (61)
$r^{(m)}$	relative power measure of m-th random fading component, (67)
X	M×1 Gaussian random column vector, (71)
E	mean vector, (72)
$C, Cov(X)$	covariance matrix of vector X, (72), (B-1)
Λ	eigenvalue matrix of C, (73)
Q	normalized modal matrix of C, (73)
diag	diagonal matrix, (74)
λ_m	m-th eigenvalue of matrix Λ, $1 \le m \le M$, (74)
V_m	m-th eigenvector of matrix Q, (74)
v_{mn}	n-th component of vector V_m, (74)
ε_m	deterministic parameter, (75)

$f_q(\xi)$	characteristic function of power-scaling q(t,f), (76)
$\chi_q(p)$	p-th cumulant of q(t,f), (77)
$\underline{E}$	transformed mean vector, (91)
ε_n	components of $\underline{E}$, (91)
$S^{(m)}$	power sum over signal pulse numbers, $1 \le m \le M$, (95)
$f_S(\xi)$	characteristic function of sum S, (96)
$f^{(m)}(\xi)$	characteristic function of $S^{(m)}$, (97)
$Y^{(m)}$	K×1 random column vector, (98)
$C^{(m)}$	covariance matrix of vector $Y^{(m)}$, (100)
$\Lambda^{(m)}$	eigenvalue matrix of $C^{(m)}$, (101)
$Q^{(m)}$	normalized modal matrix of $C^{(m)}$, (101)
$\lambda_k^{(m)}$	k-th eigenvalue of matrix $\Lambda^{(m)}$, (102)
$V_k^{(m)}$	k-th eigenvector of matrix $Q^{(m)}$, (102)
$v_{kj}^{(m)}$	j-th component of vector $V_k^{(m)}$, (102)
$\varepsilon_k^{(m)}$	transformed mean vector, (103)
h_k	auxiliary constants, $1 \le k \le K$, (111)
$\chi_S(p)$	p-th cumulant of sum S, (122)
$\chi_\gamma(p)$	p-th cumulant of output γ, (125)
$\underline{C}^{(m)}$	normalized covariance matrix, (127)
$\underline{\Lambda}^{(m)}$	eigenvalue matrix of $\underline{C}^{(m)}$, (128)
$\underline{\lambda}_k^{(m)}$	k-th eigenvalue of matrix $\underline{\Lambda}^{(m)}$, (128)
$\underline{\varepsilon}_k^{(m)}$	auxiliary constants, (137)
$\underline{C}$	common normalized covariance matrix, (140)
$\underline{\Lambda}$	eigenvalue matrix of $\underline{C}$, (141)
$\underline{\lambda}_k$	k-th eigenvalue of matrix $\underline{\Lambda}$, (141)
$\underline{h}_k$	auxiliary constants, (145)
C	covariance matrix of $g_1(t,f)$, (152)

ϕ	normalized power-scaling, (161)
σ_m^2	variance of $g_m(t,f)$, (162)
f_ϕ	characteristic function of ϕ, (163)
η_m	normalized constants, (164)
PDF	probability density function, (166), figure 2
p_ϕ	probability density function of ϕ, (166)
θ	normalized amplitude-scaling, (167)
p_θ	probability density function of θ, (168)
C,S	auxiliary functions, (172)
η	constant, (174), (179)
I_N	modified Bessel function of order N, (180)
p_γ	probability density function of processor output γ
T_{mk}	auxiliary parameters, (182)
e_{mk}	auxiliary parameters, (182)
Prob	probability, (183)
u	threshold, (183)
g_p	expansion coefficient, (183), (186)
F	scale factor, (184)
α_p	auxiliary parameters, (185)
$H_n(x)$	auxiliary function, (187)
ψ_m	fractional strengths of random components, (190)
SNR	signal-to-noise ratio E_1/N_o, in dB
Cov	normalized covariance $R_{11}(\tau,\nu)/R_{11}(0,0)$
det	determinant of matrix, (B-4)

Exact Detection Performance of Multiple-Pulse Frequency-Shift Signals in a Partially-Correlated Fading Medium with Generalized Noncentral Chi-Squared Statistics

INTRODUCTION

The transmission of a succession of time-delayed and/or frequency-shifted signal pulses through a fading medium leads to received waveforms which vary in amplitude and phase in a random fashion and with possibly complicated statistical dependencies. The evaluation of the detection capability of matched filter processing and incoherent combination in such fading situations and noise has been the subject of many studies over the years [1 - 15]. Some of these efforts have been aimed at obtaining approximations to the performance of the systems of interest, while others have yielded exact results in selected special cases.

Results for partially-correlated fading have been obtained in [3;4;6;7;9;10;11;12;14;15]. In particular, in [4;6;10;11;12;14], the characteristic functions of the decision variables have been obtained in closed form for the case of partially correlated fading of the received signal pulses with Rayleigh amplitude statistics and additive Gaussian noise. The results in [12] are approximate; however, the signal amplitude fading is not limited to Rayleigh statistics there, but rather can have a Chi distribution of an arbitrary number of degrees of freedom.

The case of independent signal fading in noise, but allowing correlated clutter, was addressed in [13]. An extension of [12]

to partially correlated signal fading in a system with normalization was solved in an approximate fashion in [14]. Finally, a case of partially correlated signal fading with Rayleigh amplitude statistics in the presence of noise and K-distributed clutter was solved exactly in [15].

Here, we will give exact results for a fading medium which has both deterministic components as well as random components with arbitrary covariance coefficients and additive Gaussian noise. Also, frequency shift keyed (FSK) signals will be allowed, with arbitrary (non-overlapping) occupancy in time, frequency space. In addition, the signal amplitude statistics will not be limited to Rayleigh, but will include the range of possibilities inherent in the noncentral Chi distribution of an arbitrary number of degrees of freedom. The end result is a closed form for the characteristic function of the decision variable that, although complicated in appearance, is amenable to efficient computer evaluation of the detection characteristics by means of a single fast Fourier transform (FFT). No matrix inverses are required. However, a novel expansion technique for the detection probability is derived which is efficient and accurate; a program incorporating this expansion is listed and exercised for several examples.

Extensions to a more general form of processor will also be solved but not evaluated. In particular, the characteristic function of the most general complex second-order form of arbitrarily-correlated complex Gaussian random variables with arbitrary means will be obtained in closed form.

RECEIVER PROCESSING FOR A SINGLE PULSE

In this section, we will describe the basic model of the single-pulse transmitted signal, the received signal with fading, the received noise, and the receiver processing. The extension to multiple signal pulses will be undertaken in the next section.

We presume that the transmitted real signal s(t) is narrowband with low-frequency complex envelope $\underline{s}(t)$ superposed on carrier frequency f_o:

$$s(t) = \mathrm{Re}\{\underline{s}(t)\ \exp(i2\pi f_o t)\}\ . \tag{1}$$

The transmitted signal energy is then

$$\tilde{E} \equiv \int dt\ [s(t)]^2 = \frac{1}{2}\int dt\ |\underline{s}(t)|^2\ , \tag{2}$$

where integrals without limits are over $(-\infty,+\infty)$. The time-bandwidth product of the single pulse complex envelope $\underline{s}(t)$ is arbitrary; thus, for example, $\underline{s}(t)$ could contain linear frequency modulation along with rectangular or Gaussian amplitude modulation.

The received signal waveform is

$$\mathrm{Re}\{r\ \underline{s}(t-t_d)\ \exp[i2\pi(f_o+f_d)t + i\theta]\}\ , \tag{3}$$

where r is a (dimensionless) amplitude scaling, t_d is a time delay, f_d is a frequency shift, and θ is a phase shift. Real random unknowns r and θ do not vary with time over the duration of pulsed signal $\underline{s}(t)$. Delay t_d and shift f_d are presumed known at the receiver. The average received signal energy is, using

(3) and (2),

$$\overline{r^2}\,\frac{1}{2}\int dt\,|\underline{s}(t)|^2 = \overline{r^2}\,\tilde{E}\ . \tag{4}$$

The received zero-mean additive noise waveform n(t) is

$$n(t) = \mathrm{Re}\{\underline{n}(t)\exp(i2\pi f_o t)\}\ , \tag{5}$$

which is presumed stationary over the duration of the signal. The spectrum of received noise n(t) in the neighborhood of frequency f_o is flat, with one-sided spectral level N_o watts/Hz.

The reference waveform employed at the receiver corresponds to the matched filter to the transmitted signal, namely

$$A\,\mathrm{Re}\{\underline{s}(t-t_d)\exp[i2\pi(f_o+f_d)t + i\phi]\}\ , \tag{6}$$

which utilizes knowledge of t_d and f_d. The local reference level A and local phase shift ϕ in (6) are irrelevant to the processing employed here; that is, the performance in terms of the receiver operating characteristics (detection probability P_D versus false alarm probability P_F) is independent of A and ϕ, at least for this case of one signal pulse.

We now define the two analytic functions

$$\begin{aligned}\alpha(t) &= r\,\underline{s}(t-t_d)\exp[i2\pi(f_o+f_d)t+i\theta] + \underline{n}(t)\exp(i2\pi f_o t)\ , \\ \beta(t) &= A\,\underline{s}(t-t_d)\exp[i2\pi(f_o+f_d)t+i\phi]\ ,\end{aligned} \tag{7}$$

which are recognized as corresponding to the received waveform and the local reference, respectively. The total output of the matched-filter squared-envelope detector to the received signal and noise waveform is then given by

$$\gamma = \left| \int dt\ \alpha(t)\ \beta^*(t) \right|^2 =$$

$$= \left| r \exp(i\theta)\ A \int dt\ |\underline{s}(t-t_d)|^2 + A \int dt\ \underline{n}(t)\ \underline{s}^*(t-t_d) \exp(-i2\pi f_d t) \right|^2$$

$$= \left| 2\ r\ A\ \tilde{E} + n_r + i\ n_i \right|^2 \equiv \left| \mu + n_r + i\ n_i \right|^2 , \tag{8}$$

where we used (2) and defined the zero-mean processor output complex noise variate

$$n_r + i\ n_i = \exp(-i\theta)\ A \int dt\ \underline{n}(t)\ \underline{s}^*(t-t_d) \exp(-i2\pi f_d t)\ . \tag{9}$$

The two covariances of the received noise complex envelope $\underline{n}(t)$ are derived in appendix A; they are given by (A-8) and (A-11) as

$$\overline{\underline{n}(t_1)\ \underline{n}(t_2)} = 0\ , \quad \overline{\underline{n}(t_1)\ \underline{n}^*(t_2)} = 2\ N_o\ \delta(t_1-t_2)\ . \tag{10}$$

Use of these relations with (9) enables us to determine the two averages

$$\overline{(n_r + i\ n_i)^2} = \overline{n_r^2} - \overline{n_i^2} + i\ 2\ \overline{n_r\ n_i} = 0\ ,$$

$$\overline{|n_r + i\ n_i|^2} = \overline{n_r^2} + \overline{n_i^2} = 2\ N_o\ A^2 \int dt\ |\underline{s}(t-t_d)|^2 = 4\ N_o\ A^2\ \tilde{E}\ , \tag{11}$$

where we also used (2). Combining the information in the two lines of (11), there follows, for the two zero-mean real variates n_r and n_i,

$$\overline{n_r^2} = \overline{n_i^2} = 2\ N_o\ A^2\ \tilde{E} \equiv \sigma^2\ , \quad \overline{n_r\ n_i} = 0\ . \tag{12}$$

These results are independent of the parameters θ, t_d, and f_d.

Since noise input $\underline{n}(t)$ is Gaussian and operation (9) is linear, real variates n_r and n_i are Gaussian with joint probability density function

$$p(n_r,n_i) = \frac{1}{2\pi\sigma^2} \exp\left(- \frac{n_r^2 + n_i^2}{2\sigma^2}\right) . \tag{13}$$

From this result, we can derive the conditional exceedance distribution function of matched filter output γ in (8), for a given value of random variable r. In terms of the two auxiliary parameters μ and σ^2 defined in (8) and (12) respectively, we have, for $u > 0$, exceedance probability

$$\text{Prob}(\gamma > u) = \text{Prob}\left((\mu + n_r)^2 + n_i^2 > u\right) =$$

$$= \iint_C dx\ dy\ \frac{1}{2\pi\sigma^2} \exp\left(- \frac{(x - \mu)^2 + y^2}{2\sigma^2}\right) =$$

$$= \frac{1}{2\pi\sigma^2} \int_{\sqrt{u}}^{\infty} dr'\ r' \int_{-\pi}^{\pi} d\theta'\ \exp\left(- \frac{r'^2 - 2\mu r'\cos\theta' + \mu^2}{2\sigma^2}\right) =$$

$$= \frac{1}{\sigma^2} \int_{\sqrt{u}}^{\infty} dr'\ r'\ \exp\left(- \frac{r'^2 + \mu^2}{2\sigma^2}\right) I_o\left(\frac{\mu r'}{\sigma^2}\right) = Q\left(\frac{\mu}{\sigma},\frac{\sqrt{u}}{\sigma}\right) , \tag{14}$$

where C is the region exterior to a circle of radius $\sqrt{u}$ centered at the origin of the x,y plane, and Q is Marcum's function [1].

The conditional probability density function of γ, for the same given value of amplitude scaling r, is

$$p(u) = - \frac{d}{du} \text{Prob}(\gamma > u) = \frac{1}{2\sigma^2} \exp\left(- \frac{u + \mu^2}{2\sigma^2}\right) I_o\left(\frac{\mu \sqrt{u}}{\sigma^2}\right) , \tag{15}$$

for u > 0. Therefore, the conditional characteristic function of the squared-envelope detector output γ, for given r, is

$$f_c(\xi) = \int du \, \exp(i\xi u) \, p(u) = \frac{1}{2\sigma^2} \int_0^\infty du \, \exp\left(i\xi u - \frac{u + \mu^2}{2\sigma^2}\right) \, I_o\left(\frac{\mu \sqrt{u}}{\sigma^2}\right)$$

$$= \frac{1}{1 - i\xi 2\sigma^2} \exp\left(\frac{i\xi\mu^2}{1 - i\xi 2\sigma^2}\right) , \qquad (16)$$

where we used [16; 6.631 4]. The two important parameters here were defined in (8) and (12) according to

$$\mu = 2 \, r \, A \, \tilde{E} \; , \quad \sigma^2 = 2 \, N_o \, A^2 \, \tilde{E} \; . \qquad (17)$$

Notice that, in this case of matched filtering, the detailed behavior of complex envelope $\underline{s}(t)$ is not relevant; only the total transmitted signal energy $\tilde{E}$ is of consequence to the conditional characteristic function (16) of output γ in (8). The results in (16) and (17) are independent of the particular value of the medium phase shift θ encountered in (3) or (7); this is due to the processing method adopted at the receiver, namely envelope detection of the matched filter output.

The cumulants of random variable γ, for given fixed amplitude scaling r, are available by developing the logarithm of (16) in a power series in $i\xi$:

$$\ln f_c(\xi) = - \ln(1 - i\xi 2\sigma^2) + i\xi\mu^2 \left(1 - i\xi 2\sigma^2\right)^{-1} =$$

$$= \sum_{j=1}^{\infty} (i\xi)^j \left(\frac{(2\sigma^2)^j}{j} + \mu^2 (2\sigma^2)^{j-1}\right) . \qquad (18)$$

There follows immediately the j-th (conditional) cumulant of γ as

$$\chi(j) = (j-1)!\ (2\sigma^2)^j \left(1 + j\,\frac{\mu^2}{2\sigma^2}\right) \quad \text{for } j \geq 1 \ . \tag{19}$$

In particular, the first two cumulants are

$$\chi(1) = 2\sigma^2 + \mu^2 \ , \qquad \chi(2) = 4\sigma^4 \left(1 + \frac{\mu^2}{\sigma^2}\right) \ . \tag{20}$$

It is convenient to define a conditional deflection criterion at the squared-envelope detector output γ as the ratio of the difference of means, with and without signal, to the noise-alone standard deviation. That is,

$$d \equiv \frac{\chi_{S+N}(1) - \chi_N(1)}{\chi_N^{1/2}(2)} = \frac{\mu^2}{2\sigma^2} = r^2\,\frac{\tilde{E}}{N_o} \ , \tag{21}$$

where we used (20) and (17). This conditional deflection depends on amplitude scaling value r, of course. However, it is independent of local reference values A and ϕ in (6), as expected, since performance measures should not depend on receiver processor absolute levels or phase shifts, at least for the case of a single pulse.

RECEIVER PROCESSING FOR MULTIPLE SIGNAL PULSES

In this section we will generalize to the case where K signal pulses are transmitted, all of which are nonoverlapping in time, frequency space; the results derived in the previous section will be used freely in the sequel. The transmitted energy in the k-th pulse is $\tilde{E}_k$, where $1 \le k \le K$. The k-th signal pulse undergoes (dimensionless) amplitude scaling r_k and phase shift θ_k in transmission through the medium to the receiver. For each k, random variables r_k and θ_k do not vary with time over the duration of the k-th individual signal pulse; this is a generalization of (3). The k-th local reference waveform employed at the receiver utilizes constant amplitude scaling A_k and phase shift ϕ_k, which can be chosen for best performance; compare (6).

It is now necessary to generalize the definitions in (7) to K pairs of analytic functions, namely

$$\alpha_k(t) = r_k\, \underline{s}_k(t-t_d)\, \exp[i2\pi(f_o+f_d)t+i\theta_k] + \underline{n}(t)\, \exp(i2\pi f_o t) \; ,$$

$$\beta_k(t) = A_k\, \underline{s}_k(t-t_d)\, \exp[i2\pi(f_o+f_d)t+i\phi_k] \; , \qquad (22)$$

which correspond to the k-th received waveform and local reference, respectively. A block diagram of the signal processing employed at the receiver is depicted in figure 1. The k-th signal complex envelope $\underline{s}_k(t)$ in (22) includes the time delay t_k and the frequency shift f_k associated with the k-th pulse, in accordance with the FSK pattern employed at the transmitter and known to the receiver. The K matched filter outputs are envelope detected, squared, and then summed to yield

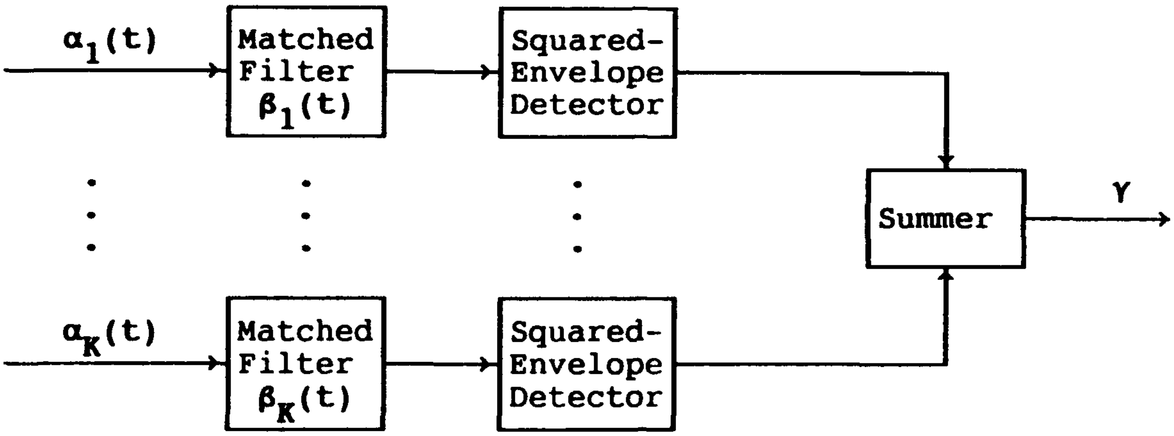

Figure 1. Block Diagram of Receiver Processing

output γ, which is compared with a threshold for purposes of declaring signal presence or absence. Observe that the matched filter $\beta_k(t)$ incorporates the local reference scaling A_k and phase shift ϕ_k and therefore accounts for (complex) weighting prior to the summation indicated in figure 1. The absolute level of weights $\{A_k\}$ does not matter and can be chosen freely; however, their values relative to each other will affect the detection capability of the processor in figure 1.

The energy in the k-th (real) reference waveform is, from (22) and (2),

$$\frac{1}{2} \int dt \; |\beta_k(t)|^2 = A_k^2 \, \tilde{E}_k \, , \tag{23}$$

where $\tilde{E}_k$ is the transmitted energy in the k-th signal pulse. In analogy to (17), we define the K parameters

$$\mu_k = 2 \, r_k \, A_k \, \tilde{E}_k \, , \quad \sigma_k^2 = 2 \, N_o \, A_k^2 \, \tilde{E}_k \quad \text{for } 1 \le k \le K \, . \tag{24}$$

The noise density level N_o is a common constant applicable to all K pulses because the received noise is stationary over the total time extent, and it is white over the entire frequency band of the K received signals.

The K signals $\{\underline{s}_k(t)\}$ are nonoverlapping in time,frequency space, (t,f). Also, the received noise is Gaussian. Therefore, the K output noise variables from the matched filters in figure 1 are statistically independent of each other; mathematically, we are using orthogonality relation

$$\int dt\ \underline{s}_k(t)\ \underline{s}_j^*(t) = 0 \quad \text{for } k \neq j\ . \tag{25}$$

This is due to nonoverlapping, in time or frequency or both, of all the component FSK pulses $\{\underline{s}_k(t)\}$, each with their individual delay,shift t_k, f_k. These observations, plus relations (8) and (16), allow us to determine the conditional characteristic function of output γ from figure 1 in the compact form

$$f_c(\xi) = \prod_{k=1}^{K}\left(1 - i\xi\ 2\sigma_k^2\right)^{-1} \exp\left[i\xi \sum_{k=1}^{K} \frac{\mu_k^2}{1 - i\xi\ 2\sigma_k^2}\right] . \tag{26}$$

It should be noticed that this result is independent of the set of phase shifts $\{\theta_k\}$ encountered in transmission; see (22) and (24). The reason for this development is the method of receiver processing adopted in figure 1, namely envelope detection of each of the matched filter outputs prior to summation. However, (26) depends significantly on the amplitude scalings $\{r_k\}$ through parameters $\{\mu_k\}$ in (24); explicitly, the alternative form of (26) is

$$f_c(\xi) = \prod_{k=1}^{K} \left(1 - i\xi\ 4\ N_o\ A_k^2\ \tilde{E}_k\right)^{-1} \exp\left[i\xi \sum_{k=1}^{K} \frac{4\ r_k^2\ A_k^2\ \tilde{E}_k^2}{1 - i\xi\ 4\ N_o\ A_k^2\ \tilde{E}_k}\right] . \tag{27}$$

Also, characteristic function (27) is independent of the receiver set of phase shifts $\{\phi_k\}$ used in local references (22), again due to the method of processing employed in figure 1. However, (27) depends upon the receiver weights $\{A_k\}$ in a complicated fashion. In order to determine a reasonable method of selecting this set of weights, we consider the following development.

The (conditional) cumulants of system output random variable γ, for a given fixed set of amplitude scalings $\{r_k\}$, are available by developing the logarithm of (26) or (27) in a power series in $i\xi$. Reference to (18) and (19) immediately yields the j-th (conditional) cumulant of γ as

$$\chi(j) = (j-1)! \sum_{k=1}^{K} \left(2\sigma_k^2\right)^j \left(1 + j\ \frac{\mu_k^2}{2\sigma_k^2}\right) \quad \text{for } j \geq 1 . \tag{28}$$

In particular, the first two cumulants of γ are

$$\chi(1) = \sum_{k=1}^{K} \left(2\sigma_k^2 + \mu_k^2\right) , \quad \chi(2) = 4 \sum_{k=1}^{K} \left(\sigma_k^4 + \sigma_k^2\ \mu_k^2\right) . \tag{29}$$

We can now define a (conditional) deflection criterion d_K for the output of the processor in figure 1, exactly as was done in (21) earlier for the single pulse case. Substitution of (29) and (24) then yields

$$d_K = \frac{\sum_{k=1}^{K} \mu_k^2}{2\left(\sum_{k=1}^{K} \sigma_k^4\right)^{\frac{1}{2}}} = \frac{\sum_{k=1}^{K} \overline{r_k^2}\, A_k^2\, \tilde{E}_k^2}{N_o \left(\sum_{k=1}^{K} A_k^4\, \tilde{E}_k^2\right)^{\frac{1}{2}}} \, . \tag{30}$$

The absolute level of receiver weights $\{A_k\}$ cancels out of (30); however, the <u>relative</u> sizes of these weights does affect the value of deflection d_K attainable.

If the average <u>power</u> scaling of the k-th pulse, $\overline{r_k^2}$, is the same for all k, that is,

$$\overline{r_k^2} = \overline{r_1^2} \quad \text{for } 2 \le k \le K \, , \tag{31}$$

then the <u>average</u> deflection of output γ in figure 1 is, from (30),

$$\overline{d_K} = \frac{\overline{r_1^2}}{N_o} \frac{\sum_{k=1}^{K} A_k^2\, \tilde{E}_k^2}{\left(\sum_{k=1}^{K} A_k^4\, \tilde{E}_k^2\right)^{\frac{1}{2}}} \le \frac{\overline{r_1^2}}{N_o} \left(\sum_{k=1}^{K} \tilde{E}_k^2\right)^{\frac{1}{2}} \, , \tag{32}$$

with equality only if all the weights A_k are equal. This result is not surprising, since there is already a built-in dependence of matched filter $\beta_k(t)$ on the transmitted signal energy $\tilde{E}_k$; namely, (23) gives this energy as $A_k^2\, \tilde{E}_k$. Thus, we have the conclusion that the "best" set of weights $\{A_k\}$, for the case of equal power scalings (31) in the medium, is uniform when local reference $\beta_k(t)$ is chosen according to (22).

More generally, the average deflection follows from (30) as

$$\overline{d_K} = \frac{\sum_{k=1}^{K} \overline{r_k^2}\, A_k^2\, \tilde{E}_k^2}{N_o \left(\sum_{k=1}^{K} A_k^4\, \tilde{E}_k^2\right)^{\frac{1}{2}}} \le \frac{1}{N_o} \left(\sum_{k=1}^{K} \overline{r_k^2}^{\,2}\, \tilde{E}_k^2\right)^{\frac{1}{2}} , \qquad (33)$$

with equality if and only if the receiver weights satisfy

$$A_k = A\, \overline{r_k^2}^{\,\frac{1}{2}} \quad \text{for } 1 \le k \le K . \qquad (34)$$

Scale factor A is arbitrary, reflecting the fact that the absolute level of set $\{A_k\}$ is irrelevant. It should be noted that the "best" receiver weights in (34) are independent of the transmitted signal energies $\{\tilde{E}_k\}$. This is again due to the fact that matched filter $\beta_k(t)$ in figure 1 and (22) already has an energy proportional to $\tilde{E}_k$; see (23).

Result (34) for the weights $\{A_k\}$ requires knowledge of the average power scaling $\overline{r_k^2}$ applied to the k-th pulse in transmission through the medium. But these average power scalings may be unknown or they may be independent of k, the particular pulse number. In this case, a reasonable choice of receiver weights is simply to take $A_k = A$ for $1 \le k \le K$. Then the conditional characteristic function in (27) of processor output γ reduces to

$$f_c(\xi) = \prod_{k=1}^{K} \left(1 - i\xi\, 4\, N_o\, A^2\, \tilde{E}_k\right)^{-1} \exp\left[i\xi 4A^2 \sum_{k=1}^{K} \frac{r_k^2\, \tilde{E}_k^2}{1 - i\xi\, 4\, N_o\, A^2\, \tilde{E}_k}\right] \qquad (35)$$

In the (usual) case where all the transmitted signal energies in the K pulses are equal, that is, $\tilde{E}_k = \tilde{E}$ for $1 \le k \le K$, then

the conditional characteristic function in (35) reduces to

$$f_c(\xi) = \left(1 - i\xi\ 4\ N_o\ A^2\ \tilde{E}\right)^{-K} \exp\left[\frac{i\xi\ 4\ A^2\ \tilde{E}^2}{1 - i\xi\ 4\ N_o\ A^2\ \tilde{E}} \sum_{k=1}^{K} r_k^2\right] . \quad (36)$$

It is now very important to notice that the <u>only</u> way that the amplitude scaling factors $\{r_k\}$ enter this characteristic function is through the single quantity (sufficient statistic)

$$S = \sum_{k=1}^{K} r_k^2 , \quad (37)$$

which is the sum of all the <u>power</u>-fading variates r_k^2 on the K transmitted pulses. This simplified result in (36) holds when the following two reasonable conditions are satisfied:

$$\tilde{E}_k = \tilde{E} , \quad A_k = A \quad \text{for } 1 \le k \le K . \quad (38)$$

That is, the transmitted signal pulse energies are all equal and the receiver scalings $\{A_k\}$ in (22) are all taken equal. This particular case will receive most of our attention.

Since the receiver absolute scaling A in (38) is under our control and does not affect detectability performance, we take, for notational convenience and without loss of generality,

$$A^2 = \frac{1}{2\ N_o\ \tilde{E}} . \quad (39)$$

Then the conditional characteristic function in (36) of processor output γ in figure 1 simplifies to

$$f_c(\xi) = (1 - i2\xi)^{-K} \exp\left[\frac{i\xi}{1 - i2\xi} \frac{2\tilde{E}}{N_o} S\right] , \quad (40)$$

where we used (37).

Another important observation must be made at this point. Suppose the characteristic function of random variable S, defined by (37), is known; that is,

$$f_S(\xi) = \overline{\exp(i\xi S)} \tag{41}$$

is known. Then the unconditional characteristic function $f_\gamma(\xi)$ of processor output γ in figure 1 is obtained by averaging (40) over S and using (41); there follows

$$f_\gamma(\xi) = (1 - i2\xi)^{-K}\, f_S\!\left(\frac{\xi}{1 - i2\xi}\,\frac{2\tilde{E}}{N_o}\right) . \tag{42}$$

This compact form for the unconditional characteristic function of processor output γ depends critically upon being able to obtain the characteristic function $f_S(\xi)$ in (41) of the power-fading summation S defined in (37). Therefore, effort can be concentrated on attempting to determine (41), either exactly or by use of several low-order moments of S; see [17] for example. Of course, in the process, the amount of partial correlation between individual pairs of pulse power-fading variates $\{r_k^2\}$ in sum S will come into consideration. In any event, the characteristic function of sum S in (37) is the major item of interest; given this quantity, the processor output γ in figure 1 is completely characterized in terms of its characteristic function (42), at least when conditions (38) are satisfied.

The more general conditional characteristic function in (27), with arbitrary $\{\tilde{E}_k\}$ and $\{A_k\}$, will be treated later, after we have introduced a detailed model of the amplitude fadings $\{r_k\}$.

CHARACTERIZATION OF FADING

We are interested in obtaining characteristic function $f_S(\xi)$ of sum S of power-fading variates $\{r_k^2\}$ in (37); its use in (42) will then yield the desired characteristic function $f_\gamma(\xi)$ of receiver output γ in figure 1. However, we must first concentrate on characterizing the fading which gives rise to sum S.

COVARIANCE COEFFICIENTS OF POWER-FADING VARIATES $\{q_k\}$

Define the k-th power-fading variate q_k as the square of the amplitude-fading variate r_k in (22):

$$q_k = r_k^2 \quad \text{for } 1 \le k \le K \, . \tag{43}$$

The model of fading that we consider here is that power scaling q_k is a sample of a continuous fading process q(t,f), namely

$$q_k = q(t_k, f_k) \quad \text{for } 1 \le k \le K \, , \tag{44}$$

where two-dimensional function of time t and frequency f,

$$q(t,f) = \sum_{m=1}^{M} \left[c_m(t,f) + g_m(t,f) \right]^2 \, . \tag{45}$$

Sampling time t_k and frequency f_k correspond to the time-delay location and frequency-shift location, respectively, of the k-th FSK complex envelope signal $\underline{s}_k(t)$ employed in (22).

The number of fading components in model (45) is M. Each component contains a deterministic part $c_m(t,f)$ and a stationary zero-mean Gaussian part $g_m(t,f)$. Without loss of generality, the

nonrandom part satisfies

$$c_m(t,f) \geq 0 \quad \text{for all } m,\ t,\ f\ . \tag{46}$$

Random parts $\{g_m(t,f)\}$ of the joint Gaussian processes are stationary in both t and f, with covariances

$$\overline{g_m(t,f)\ g_n(t-\tau,f-\nu)} = R_{mn}(\tau,\nu) \quad \text{for } 1 \leq m,\ n \leq M\ . \tag{47}$$

(It is possible to generalize to nonstationary Gaussian processes $\{g_m(t,f)\}$; however, each covariance R_{mn} would then be a function of the four variables τ,ν,t,f instead of just differences τ,ν.) From (47), we have property

$$R_{nm}(-\tau,-\nu) = R_{mn}(\tau,\nu)\ . \tag{48}$$

Fading model (44) - (45) does not constitute a multipath medium but does mimic its net effect. Every transmitted pulse $\underline{s}_k(t)$ undergoes just one common time delay t_d and frequency shift f_d, as indicated in (22), which are known and utilized by the receiver. Rather, each signal pulse simply undergoes a different phase shift θ_k and amplitude scaling r_k in (22), the latter of which is characterized through power scaling q_k in (43) - (45).

The latter random variable, q_k, is more general than non-central Chi-squared because the random components $\{g_m(t,f)\}$ in (45) can have unequal variances $\{R_{mm}(0,0)\}$ and can be correlated with each other; that is, we allow $R_{mn}(\tau,\nu) \neq 0$ for $m \neq n$.

The mean of power-fading variate q_k is given by

$$\overline{q_k} = \sum_{m=1}^{M} \left[c_m^2(t_k,f_k) + R_{mm}(0,0)\right] \quad \text{for } 1 \leq k \leq K\ , \tag{49}$$

where we used (44), (45), and (47). This quantity is independent of pulse number k if all M deterministic components are zero, or if they do not depend on t or f; that is, if $c_m(t,f)$ is independent of t and f for $1 \le m \le M$. Since $\tilde{E}_k$ is the <u>transmitted</u> signal energy in the k-th pulse, then

$$\tilde{E}_k \, c_m^2(t_k,f_k) \equiv D_{km} = \text{deterministic \underline{received} signal energy in m-th component of k-th pulse,} \tag{50}$$

$$\tilde{E}_k \sum_{m=1}^{M} c_m^2(t_k,f_k) = \sum_{m=1}^{M} D_{km} \equiv D_k = \text{deterministic received signal energy in k-th pulse ,} \tag{51}$$

$$\tilde{E}_k \, R_{mm}(0,0) \equiv E_{km} = \text{average random \underline{received} signal energy in m-th component of k-th pulse,} \tag{52}$$

$$\tilde{E}_k \sum_{m=1}^{M} R_{mm}(0,0) = \sum_{m=1}^{M} E_{km} \equiv E_k = \text{average random received signal energy in k-th pulse .} \tag{53}$$

The alternating (zero-mean) component of power scaling q_k is then given by (44), (45), and (49) as

$$\tilde{q}_k = q_k - \overline{q_k} =$$

$$= \sum_{m=1}^{M} \left[2\, c_m(t_k,f_k)\, g_m(t_k,f_k) + g_m^2(t_k,f_k) - R_{mm}(0,0) \right] . \tag{54}$$

The covariance between a pair of power-fading variates is

$$\tilde{R}_{kj} = \overline{\tilde{q}_k \, \tilde{q}_j} \; , \tag{55}$$

while the covariance coefficient between q_k and q_j is

$$\rho_{kj} = \frac{\tilde{R}_{kj}}{\left(\tilde{R}_{kk}\ \tilde{R}_{jj}\right)^{\frac{1}{2}}} \ . \tag{56}$$

Thus, the fundamental calculation required for determination of covariance coefficient ρ_{kj} is $\tilde{R}_{kj}$ as defined by (55) in conjunction with (54).

When we substitute (54) in (55) and use the fact that $\{g_m(t,f)\}$ are zero-mean joint Gaussian processes, the first-order and third-order moments involving $\{g_m(t,f)\}$ are zero, while the fourth-order moment can be broken down into a sum of products of second-order moments, leading to covariance

$$\tilde{R}_{kj} = 2\sum_{m,n=1}^{M} R_{mn}^2(t_k-t_j, f_k-f_j) + \\ + 4\sum_{m,n=1}^{M} R_{mn}(t_k-t_j, f_k-f_j)\ c_m(t_k,f_k)\ c_n(t_j,f_j) \ . \tag{57}$$

In particular, the variance of power-fading variate q_k is

$$\tilde{R}_{kk} = \overline{\tilde{q}_k^2} = 2\sum_{m,n=1}^{M} R_{mn}^2(0,0) + \\ + 4\sum_{m,n=1}^{M} R_{mn}(0,0)\ c_m(t_k,f_k)\ c_n(t_k,f_k) \ . \tag{58}$$

The covariance coefficient ρ_{kj} between q_k and q_j is obtained upon substitution of general results (57) and (58) into (56).

A large number of special cases can be obtained from the general formulation of fading in (45) and (47). Below, we list seven special cases that will occupy most of our attention in the remainder of this report, and which will consistently be referred to according to their number in the sequel.

SPECIAL CASE 1: Zero means $\{c_m(t,f)\}$

$$c_m(t,f) = 0 \quad \text{for } 1 \le m \le M \; . \tag{59}$$

$$\tilde{R}_{kj} = 2 \sum_{m,n=1}^{M} R_{mn}^2(t_k-t_j, f_k-f_j) \; ,$$

$$\tilde{R}_{kk} = 2 \sum_{m,n=1}^{M} R_{mn}^2(0,0) \; . \tag{60}$$

SPECIAL CASE 2: Uncorrelated components $\{g_m(t,f)\}$

$$R_{mn}(\tau,\nu) = R_{mm}(\tau,\nu)\, \delta_{mn} \quad \text{for } 1 \le m,\ n \le M \; . \tag{61}$$

$$\tilde{R}_{kj} = 2 \sum_{m=1}^{M} R_{mm}^2(t_k-t_j, f_k-f_j) +$$

$$+ 4 \sum_{m=1}^{M} R_{mm}(t_k-t_j, f_k-f_j)\, c_m(t_k,f_k)\, c_m(t_j,f_j) \; ,$$

$$\tilde{R}_{kk} = 2 \sum_{m=1}^{M} R_{mm}^2(0,0) + 4 \sum_{m=1}^{M} R_{mm}(0,0)\, c_m^2(t_k,f_k) \; . \tag{62}$$

It is very important to note that correlated fading still exists between the received signal pulses; that is, covariance $\tilde{R}_{kj}$ in (62) is <u>not</u> zero, despite uncorrelated property (61) in case 2.

SPECIAL CASE 3: Zero means and uncorrelated components

$$\tilde{R}_{kj} = 2 \sum_{m=1}^{M} R_{mm}^2(t_k-t_j, f_k-f_j) ,$$

$$\tilde{R}_{kk} = 2 \sum_{m=1}^{M} R_{mm}^2(0,0) ,$$

$$\rho_{kj} = \frac{\sum_{m=1}^{M} R_{mm}^2(t_k-t_j, f_k-f_j)}{\sum_{m=1}^{M} R_{mm}^2(0,0)} . \tag{63}$$

SPECIAL CASE 4: Uncorrelated components with identical covariances $\{R_{mn}(\tau,\nu)\}$

$$R_{mn}(\tau,\nu) = R_{11}(\tau,\nu)\ \delta_{mn} \quad \text{for all } m,\ n . \tag{64}$$

$$\tilde{R}_{kj} = 2M\ R_{11}^2(t_k-t_j, f_k-f_j) +$$

$$+ 4\ R_{11}(t_k-t_j, f_k-f_j) \sum_{m=1}^{M} c_m(t_k,f_k)\ c_m(t_j,f_j) ,$$

$$\tilde{R}_{kk} = 2M\ R_{11}^2(0,0) + 4\ R_{11}(0,0) \sum_{m=1}^{M} c_m^2(t_k,f_k) . \tag{65}$$

SPECIAL CASE 5: Zero means and uncorrelated components with identical covariances

$$\tilde{R}_{kj} = 2M\, R_{11}^2(t_k-t_j, f_k-f_j) \; ,$$

$$\tilde{R}_{kk} = 2M\, R_{11}^2(0,0) \; ,$$

$$\rho_{kj} = \left(\frac{R_{11}(t_k-t_j, f_k-f_j)}{R_{11}(0,0)}\right)^2 \; . \tag{66}$$

In this special case, the covariance coefficient between power-fading variates $q_k = q(t_k,f_k)$ and $q_j = q(t_j,f_j)$ is the square of the covariance coefficient between the amplitude-fading variates $g_m(t_k,f_k)$ and $g_m(t_j,f_j)$; see (47). Due to the identical covariances, (64), this latter covariance coefficient is the same for every m in the range $1 \le m \le M$.

SPECIAL CASE 6: Uncorrelated components with proportional covariances

This is a generalization of special case 4; it allows the random components $\{g_m(t,f)\}$ in (45) to have different strengths, as might be encountered in a fading medium. (Note: In the following, the constant $r^{(m)}$ is the average relative power measure of the m-th random fading component; it must not be confused with the random variable r_k which is the amplitude-scaling on the k-th signal pulse in (22).)

$$R_{mn}(\tau,\nu) = R_{11}(\tau,\nu)\; r^{(m)}\; \delta_{mn}\;, \qquad r^{(1)} = 1\;. \tag{67}$$

$$\tilde{R}_{kj} = 2\; R_{11}^2(t_k-t_j,f_k-f_j) \sum_{m=1}^{M} r^{(m)^2} +$$

$$+\; 4\; R_{11}(t_k-t_j,f_k-f_j) \sum_{m=1}^{M} r^{(m)}\; c_m(t_k,f_k)\; c_m(t_j,f_j)\;,$$

$$\tilde{R}_{kk} = 2\; R_{11}^2(0,0) \sum_{m=1}^{M} r^{(m)^2} + 4\; R_{11}(0,0) \sum_{m=1}^{M} r^{(m)}\; c_m^2(t_k,f_k)\;. \tag{68}$$

SPECIAL CASE 7: Zero means and uncorrelated components with proportional covariances

Setting the means $\{c_m(t,f)\}$ in (68) to zero,

$$\tilde{R}_{kj} = 2\; R_{11}^2(t_k-t_j,f_k-f_j) \sum_{m=1}^{M} r^{(m)^2},$$

$$\tilde{R}_{kk} = 2\; R_{11}^2(0,0) \sum_{m=1}^{M} r^{(m)^2},$$

$$\rho_{kj} = \left(\frac{R_{11}(t_k-t_j,f_k-f_j)}{R_{11}(0,0)}\right)^2\;. \tag{69}$$

Notice, in this special case, that $\{\rho_{kj}\}$ are independent of the values of $\{r^{(m)}\}$, the relative power measures of the M random fading components.

CHARACTERISTIC FUNCTION OF POWER-FADING VARIATE q(t,f)

The instantaneous power-fading process is, from (45),

$$q(t,f) = \sum_{m=1}^{M} \left[c_m(t,f) + g_m(t,f)\right]^2 = X^T X , \tag{70}$$

where M×1 Gaussian random column vector

$$X = \left[c_1(t,f) + g_1(t,f) \quad . . . \quad c_M(t,f) + g_M(t,f)\right]^T . \tag{71}$$

We now appeal directly to the very general result derived in appendix B, for the characteristic function of a quadratic form and linear form, and identify the quantities there according to

$$N = M , \quad B = I , \quad A = 0 , \quad E = \left[c_1(t,f) \quad . . . \quad c_M(t,f)\right]^T ,$$

$$C = \mathrm{Cov}(X) = \left[\overline{g_m(t,f)\ g_n(t,f)}\right]_1^M = \left[R_{mn}(0,0)\right]_1^M . \tag{72}$$

It is important to note that M×M symmetric covariance matrix C is <u>not</u> a function of t,f, under the stationarity assumption (47) for all M random processes $\{g_m(t,f)\}$.

According to (B-10), we must solve the standard characteristic-value matrix equation

$$C\ Q = Q\ \Lambda , \tag{73}$$

for M×M eigenvalue matrix Λ and normalized modal matrix Q, where

$$\Lambda = \mathrm{diag}\left[\lambda_1 \ldots \lambda_M\right] , \quad Q = \left[V_1 \ldots V_M\right] , \quad V_m = \left[v_{m1} \ldots v_{mM}\right]^T , \tag{74}$$

and M×1 column vector V_m is the m-th eigenvector with components

$\{v_{mn}\}$, $1 \le n \le M$. Then (B-18) with (74) and (72) yields deterministic parameters

$$\varepsilon_m = V_m^T E = \sum_{n=1}^{M} v_{mn} c_n(t,f) \equiv \varepsilon_m(t,f) , \quad \alpha_m = 0 , \tag{75}$$

for $1 \le m \le M$. We can now use (B-20) to obtain the characteristic function of power-fading variate $q(t,f)$ as

$$f_q(\xi) = \left[\prod_{m=1}^{M}\left(1 - i2\xi\lambda_m\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{m=1}^{M} \frac{\varepsilon_m^2(t,f)}{1 - i2\xi\lambda_m}\right] . \tag{76}$$

The eigenvectors $\{V_m\}$ and nonzero means $\{c_m(t,f)\}$ enter this result through the terms $\{\varepsilon_m(t,f)\}$ defined in (75). This general result will be simplified, below, to the seven special cases that were presented earlier in (59) - (69).

By expanding the logarithm of general result (76) in a power series in $i\xi$, the cumulants of $q(t,f)$ are found to be

$$\chi_q(p) = 2^{p-1}\,(p-1)! \sum_{m=1}^{M} \left(\lambda_m + p\, \varepsilon_m^2(t,f)\right) \lambda_m^{p-1} \quad \text{for } p \ge 1 . \tag{77}$$

For $p = 1$ and $p = 2$, these may be verified to equal the results in (49) and (58); the pertinent matrix manipulations are indicated in appendix B for the more general case where linear form $A \ne 0$. The quantities $\{\lambda_m\}$ are the eigenvalues of covariance matrix C in (72), while parameters $\{\varepsilon_m(t,f)\}$ are given by (75) in terms of eigenvectors $\{V_m\}$ of matrix C and mean vector $E = E(t,f)$ in (72).

SPECIAL CASE 1: Zero means; see (59)

Since $c_m(t,f) = 0$, then $\varepsilon_m(t,f) = 0$ from (75), and the characteristic function in (76) reduces to

$$f_q(\xi) = \left[\prod_{m=1}^{M}\left(1 - i2\xi\lambda_m\right)\right]^{-\frac{1}{2}} . \tag{78}$$

SPECIAL CASE 2: Uncorrelated components; see (61)

Here, from (72), we have

$$C = \text{diag}[R_{11}(0,0) \ . \ . \ . \ R_{MM}(0,0] \ . \tag{79}$$

Then, the eigenvalue matrix Λ in (73) must be identical to C, in which case (73) becomes C Q = Q C, where C is diagonal. This forces Q to be diagonal also; finally, normalization of Q yields Q = I. There follows, from (74) - (76),

$$v_{mn} = \delta_{mn} \ , \qquad \varepsilon_m = c_m(t,f) \ , \tag{80}$$

and characteristic function

$$f_q(\xi) = \left[\prod_{m=1}^{M}\left(1 - i2\xi R_{mm}(0,0)\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{m=1}^{M} \frac{c_m^2(t,f)}{1-i2\xi R_{mm}(0,0)}\right] . \tag{81}$$

SPECIAL CASE 3: Zero means and uncorrelated components

Here, we simply set the constants in (81) equal to zero:

$$f_q(\xi) = \left[\prod_{m=1}^{M}\left(1 - i2\xi \ R_{mm}(0,0)\right)\right]^{-\frac{1}{2}} . \tag{82}$$

SPECIAL CASE 4: Uncorrelated components with identical covariances; see (64)

Now we use (64) on result (81) to get

$$f_q(\xi) = \left(1-i2\xi R_{11}(0,0)\right)^{-M/2} \exp\left[\frac{i\xi}{1-i2\xi R_{11}(0,0)} \sum_{m=1}^{M} c_m^2(t,f)\right]. \quad (83)$$

It is important to observe in this case that only the <u>sum</u> of the squares of the means, $\{c_m(t,f)\}$, matters in so far as the characteristic function of power scaling $q(t,f)$ is concerned.

SPECIAL CASE 5: Zero means and uncorrelated components with identical covariances

Upon setting the means in (83) equal to zero, there follows

$$f_q(\xi) = \left(1 - i2\xi \, R_{11}(0,0)\right)^{-M/2}. \quad (84)$$

This special case is equivalent to [12; (A-21)] when we take the parameter m there equal to M/2.

SPECIAL CASE 6: Uncorrelated components with proportional covariances; see (67)

From (67), there follows

$$R_{mn}(0,0) = R_{11}(0,0)\; r^{(m)}\; \delta_{mn} \;, \qquad r^{(1)} = 1 \;. \tag{85}$$

Use of this relation on (81) yields characteristic function

$$f_q(\xi) = \left[\prod_{m=1}^{M}\left(1-i2\xi R_{11}(0,0)r^{(m)}\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{m=1}^{M} \frac{c_m^2(t,f)}{1-i2\xi R_{11}(0,0)r^{(m)}}\right]. \tag{86}$$

The cumulants of power scaling q(t,f) can be obtained by expanding the logarithm of characteristic function (86) in a power series in $i\xi$; there follows

$$\frac{\chi_q(p)}{(p-1)!\; 2^{p-1}} = R_{11}^{p}(0,0) \sum_{m=1}^{M} r^{(m)p} + p\, R_{11}^{p-1}(0,0) \sum_{m=1}^{M} r^{(m)p-1}\; c_m^2(t,f) \quad \text{for } p \geq 1 \;. \tag{87}$$

SPECIAL CASE 7: Zero means and uncorrelated components with proportional covariances

Setting the means in (86) to zero, we have

$$f_q(\xi) = \left[\prod_{m=1}^{M}\left(1 - i2\xi\; R_{11}(0,0)\; r^{(m)}\right)\right]^{-\frac{1}{2}} . \tag{88}$$

Actual numerical examples displaying typical probability density functions of power-scaling random variable q, for a variety of different parameter values, will be presented in a later section.

CHARACTERISTIC FUNCTION OF POWER-SUM VARIATE S

The power-sum variate S is given in (37) and (43) as the sum of the power scalings $q_k = q(t_k, f_k)$ over the total of K pulses transmitted and received:

$$S = \sum_{k=1}^{K} q(t_k, f_k) = \sum_{k=1}^{K} \sum_{m=1}^{M} \left[c_m(t_k, f_k) + g_m(t_k, f_k)\right]^2 , \qquad (89)$$

where we used (44) and (45). This double sum can be recognized as a quadratic form in KM correlated nonzero-mean Gaussian random variables. Therefore, the general approach in appendix B can be used directly to find the characteristic function of S for <u>any</u> statistical dependencies between the M zero-mean Gaussian processes $\{g_m(t,f)\}$ and <u>any</u> layout of the K points $\{t_k, f_k\}$ in the time,frequency plane. In order to realize form (89), we identify variate $q = S$, constant $N = KM$, and matrices $B = I$, $A = 0$ in appendix B. Mean column vector E in (B-1) is KM×1, while symmetric covariance matrix C is KM×KM. Then (B-10) becomes the standard characteristic-value matrix equation [18; section 1.13]

$$C\,Q = Q\,\Lambda \ , \quad C = \left[\overline{g_m(t_k, f_k)\; g_n(t_j, f_j)}\right] = \left[R_{mn}(t_k - t_j, f_k - f_j)\right], \qquad (90)$$

where KM×KM modal matrix $Q = [V_1 \; . \; . \; . \; V_{KM}]$ from (B-11). Also, there follows from (B-18),

$$\underline{E} = Q^T E \ , \quad \varepsilon_n = V_n^T E \quad \text{for } 1 \le n \le KM \ ; \quad \alpha_n = 0 \ . \qquad (91)$$

The characteristic function of sum S in (89) then follows from (B-20) as

$$f_S(\xi) = \left[\prod_{n=1}^{KM}\left(1 - i2\xi\lambda_n\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{n=1}^{KM} \frac{\varepsilon_n^2}{1 - i2\xi\lambda_n}\right] . \tag{92}$$

This general result will now be specialized to the seven cases of particular interest here.

SPECIAL CASE 1: Zero means; see (59)

When $c_m(t,f) = 0$ for $1 \le m \le M$, then mean vector E is zero, meaning that its components $\{\varepsilon_n\}$ in (91) are zero. The result in (92) then reduces to

$$f_S(\xi) = \left[\prod_{n=1}^{KM}\left(1 - i2\xi\lambda_n\right)\right]^{-\frac{1}{2}} . \tag{93}$$

Only the KM eigenvalues of KM×KM covariance matrix C of the KM random variables $\{g_m(t_k,f_k)\}$ need to be evaluated in this case.

SPECIAL CASE 2: Uncorrelated components; see (61)

From this point on, we shall be interested in the more restricted case where component Gaussian process $g_m(t,f)$ is uncorrelated with (independent of) process $g_n(t',f')$ for $m \ne n$, regardless of the values of t,f and t',f'; this case was also considered earlier in (61). This will probably encompass most situations of practical interest; furthermore, it still allows for correlated fading between the K received signal pulses. That is, covariance $\tilde{R}_{jk}$ in (62), between power scalings q_k and q_j,

need not be zero, despite uncorrelated property (61) between components $\{g_m(t,f)\}$ in this special case 2.

In this case, a simpler and more direct approach is possible and is adopted. We begin by expressing (89) as

$$S = \sum_{m=1}^{M} S^{(m)} , \tag{94}$$

where we have defined the M random variables

$$S^{(m)} = \sum_{k=1}^{K} \left[c_m(t_k,f_k) + g_m(t_k,f_k)\right]^2 \quad \text{for } 1 \le m \le M . \tag{95}$$

This superscript m notation is adopted in order to readily distinguish the fading component numbers, $1 \le m \le M$, from the time,frequency signal pulse numbers, $1 \le k \le K$; see (45) versus figure 1.

Now, it is important to observe, for this special case 2, that these latter M random variables $\{S^{(m)}\}$ are statistically independent of each other, allowing us to develop the characteristic function of sum S in (94) in the finite product form

$$f_S(\xi) = \prod_{m=1}^{M} f^{(m)}(\xi) , \tag{96}$$

where the m-th characteristic function is given by average

$$f^{(m)}(\xi) = \overline{\exp\left(i\xi S^{(m)}\right)} , \tag{97}$$

in terms of quadratic sum $S^{(m)}$ in (95).

In order to ascertain the characteristic function of sum $S^{(m)}$, we define K×1 Gaussian column vector $Y^{(m)}$ according to

$$Y^{(m)} = \left[c_m(t_1,f_1) + g_m(t_1,f_1) \ . \ . \ . \ c_m(t_K,f_K) + g_m(t_K,f_K)\right]^T . \quad (98)$$

Then, $S^{(m)}$ in (95) can be written in the quadratic form

$$S^{(m)} = Y^{(m)^T} Y^{(m)} . \quad (99)$$

We now appeal to the general results in appendix B and identify the quantities there according to $q = S^{(m)}$, $N = K$, and

$$B = I \ , \quad A = 0 \ , \quad E = E^{(m)} = \overline{Y^{(m)}} = \left[c_m(t_1,f_1) \ . \ . \ . \ c_m(t_K,f_K)\right]^T ,$$

$$C = C^{(m)} = \mathrm{Cov}\left(Y^{(m)}\right) = \left[\overline{g_m(t_k,f_k) \ g_m(t_j,f_j)}\right]_1^K = \left[R_{mm}(t_k-t_j,f_k-f_j)\right]_1^K \quad (100)$$

The K diagonal elements of matrix $C^{(m)}$ are all equal to $R_{mm}(0,0)$.

According to (B-10), we must now solve, for each value of m in the range $1 \le m \le M$, the K×K characteristic-value equation

$$C^{(m)} Q^{(m)} = Q^{(m)} \Lambda^{(m)} , \quad (101)$$

for K×K eigenvalue matrix $\Lambda^{(m)}$ and corresponding K×K normalized modal matrix $Q^{(m)}$, where

$$\Lambda^{(m)} = \mathrm{diag}\left[\lambda_1^{(m)} \ . \ . \ . \ \lambda_K^{(m)}\right] , \quad Q^{(m)} = \left[v_1^{(m)} \ . \ . \ . \ v_K^{(m)}\right] ,$$

$$v_k^{(m)} = \left[v_{k1}^{(m)} \ . \ . \ . \ v_{kK}^{(m)}\right]^T , \quad (102)$$

and K×1 vector $v_k^{(m)}$ is the k-th eigenvector with components $\{v_{kj}^{(m)}\}$, $1 \le j \le K$. Then, (B-18) with (102) and (100) yields

$$\varepsilon_k^{(m)} = v_k^{(m)^T} E^{(m)} = \sum_{j=1}^{K} v_{kj}^{(m)} c_m(t_j,f_j) , \quad \alpha_k^{(m)} = 0 . \quad (103)$$

We can now use (B-20) to obtain the characteristic function of variate $S^{(m)}$ in (99) as

$$f^{(m)}(\xi) = \left[\prod_{k=1}^{K}\left(1 - i2\xi\lambda_k^{(m)}\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{k=1}^{K} \frac{{\varepsilon_k^{(m)}}^2}{1 - i2\xi\lambda_k^{(m)}}\right] . \quad (104)$$

The desired characteristic function of power-sum variate S in (89) is then given by (96), applied to (104):

$$f_S(\xi) = \left[\prod_{m=1}^{M}\prod_{k=1}^{K}\left(1 - i2\xi\lambda_k^{(m)}\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{m=1}^{M}\sum_{k=1}^{K} \frac{{\varepsilon_k^{(m)}}^2}{1 - i2\xi\lambda_k^{(m)}}\right] . \quad (105)$$

This characteristic function of sum S is a very general result, applicable to the case of uncorrelated components $\{g_m(t,f)\}$; however, it does require the solution of M matrix equations of the form of (101), each matrix being of size K×K. Nevertheless, this approach is significantly simpler than solving the one large KM×KM matrix equation (90).

SPECIAL CASE 3: Zero means and uncorrelated components

When the means $\{c_m(t,f)\}$ are zero, (103) yields $\varepsilon_k^{(m)} = 0$ and the characteristic function in (105) reduces to just the product

$$f_S(\xi) = \left[\prod_{m=1}^{M}\prod_{k=1}^{K}\left(1 - i2\xi\lambda_k^{(m)}\right)\right]^{-\frac{1}{2}} . \quad (106)$$

Eigenvalues $\lambda_1^{(m)}, \ldots, \lambda_K^{(m)}$ are found from K×K covariance matrix $C^{(m)}$ in (100); this solution must be repeated for $1 \le m \le M$.

SPECIAL CASE 4: Uncorrelated components with identical covariances; see (64)

Here, covariance matrix $C^{(m)}$ in (100) is independent of m; in particular, we now have the single K×K covariance matrix

$$C = \left[R_{11}(t_k - t_j, f_k - f_j)\right]_1^K . \tag{107}$$

This leads to solution matrices $Q^{(m)}$ and $\Lambda^{(m)}$ in (101) which are also independent of m; that is, (101) becomes the single K×K characteristic-value matrix equation

$$C\,Q = Q\,\Lambda . \tag{108}$$

Thus, the eigenvalues and eigenvectors in (102) are independent of m. However, the constants $\{\varepsilon_k^{(m)}\}$ in (103) still depend on m through their dependencies on means $\{c_m(t,f)\}$; that is, from (103),

$$\varepsilon_k^{(m)} = V_k^T\, E^{(m)} = \sum_{j=1}^{K} v_{kj}\, c_m(t_j, f_j) . \tag{109}$$

The collection of all these conclusions enables us to reduce the characteristic function (105) of S to the compact closed form

$$f_S(\xi) = \left[\prod_{k=1}^{K}\left(1 - i2\xi\lambda_k\right)\right]^{-M/2} \exp\left[i\xi \sum_{k=1}^{K} \frac{h_k}{1 - i2\xi\lambda_k}\right] , \tag{110}$$

where we defined the K constants

$$h_k = \sum_{m=1}^{M} \varepsilon_k^{(m)2} \quad \text{for } 1 \le k \le K . \tag{111}$$

If the number, M, of components in fading model (45) happens to be even, the calculation of (110) does not involve a square root, thereby further simplifying its numerical evaluation. The collapsing of the KM constants $\{\varepsilon_k^{(m)}\}$ into a smaller set of K constants, by means of the sums of squares in (111), is the analog of the result in (83) for an individual fading variate q(t,f) in special case 4.

As a special subcase here, suppose that the M deterministic components $\{c_m(t,f)\}$ in (45) are independent of t and f; that is,

$$c_m(t_k,f_k) = c_m \quad \text{for } 1 \le k \le K . \tag{112}$$

This means that the constants are independent of the locations $\{t_k,f_k\}$ of the signal pulses in the t,f plane, although they can still depend on the component number m. Then, there follows from (109) and (111),

$$\varepsilon_k^{(m)} = c_m \sum_{j=1}^{K} v_{kj} , \qquad h_k = \left(\sum_{m=1}^{M} c_m^2\right) \left(\sum_{j=1}^{K} v_{kj}\right)^2 . \tag{113}$$

That is, the characteristic function of sum S in (110) depends on the constants $\{c_m\}$ only through their sum of squares. This latter property holds true only when special subcase (112) is valid.

SPECIAL CASE 5: Zero means and uncorrelated components with identical covariances

Upon setting the means $\{c_m(t,f)\}$ to zero, it is seen from (109) and (111) that characteristic function (110) reduces to

$$f_S(\xi) = \left[\prod_{k=1}^{K}\left(1 - i2\xi\lambda_k\right)\right]^{-M/2} . \tag{114}$$

Now, only the K eigenvalues $\{\lambda_k\}$ of K×K covariance matrix C in (107) need to be evaluated. This result in (114) is essentially identical with [10; (D-14)]. For the special case of M = 2, this same result for the characteristic function of S can be shown to follow from [11; (20)].

SPECIAL CASE 6: Uncorrelated components with proportional covariances; see (67)

From (67) and (100), we find that K×K covariance matrix

$$C^{(m)} = r^{(m)}\, C \;, \quad \text{where} \quad C = \left[R_{11}(t_k - t_j, f_k - f_j)\right]_1^K , \tag{115}$$

and $r^{(1)} = 1$. Then (101) yields

$$Q^{(m)} = Q \;, \quad \Lambda^{(m)} = r^{(m)}\, \Lambda \;, \tag{116}$$

where Q and Λ are the solutions to the single K×K characteristic-value matrix equation

$$C\, Q = Q\, \Lambda \, . \tag{117}$$

We now refer to (103) to get constants

$$\varepsilon_k^{(m)} = V_k^T\, E^{(m)} = \sum_{j=1}^{K} v_{kj}\; c_m(t_j, f_j) \; . \tag{118}$$

Then (104) yields the characteristic function of $S^{(m)}$ as

$$f^{(m)}(\xi) = \left[\prod_{k=1}^{K}\left(1 - i2\xi\lambda_k r^{(m)}\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{k=1}^{K} \frac{\varepsilon_k^{(m)^2}}{1 - i2\xi\lambda_k r^{(m)}}\right] . \tag{119}$$

Finally, (96) gives the characteristic function of sum S as

$$f_S(\xi) = \left[\prod_{m=1}^{M}\prod_{k=1}^{K}\left(1 - i2\xi\; \lambda_k\; r^{(m)}\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \sum_{m=1}^{M}\sum_{k=1}^{K} \frac{\varepsilon_k^{(m)^2}}{1 - i2\xi\; \lambda_k\; r^{(m)}}\right] \; . \tag{120}$$

Here, even if the M deterministic components $\{c_m(t,f)\}$ were independent of t and f, as previously assumed in (112), the characteristic function of S would not depend simply on the sum of squares $\sum_m c_m^2$. To see this, substitute the left member of (113) in (120).

SPECIAL CASE 7: Zero means and uncorrelated components with proportional covariances

Set the means $\{c_m(t,f)\}$ to zero in (118), at which point the characteristic function for S in (120) simplifies to

$$f_S(\xi) = \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi\, \lambda_k\, r^{(m)}\right)\right]^{-\frac{1}{2}} . \tag{121}$$

Eigenvalues $\{\lambda_k\}$ are the diagonal elements of eigenvalue matrix Λ in $K \times K$ matrix equation (117), where covariance matrix C is given by (115).

CUMULANTS OF SUM S

For some purposes, such as approximating the probability density function or exceedance distribution function of S, the cumulants are useful; see, for example, [17]. By expanding the natural logarithm of the general characteristic function in (92) in a power series in $i\xi$, the cumulants of S are readily found:

$$\chi_S(p) = (p-1)!\; 2^{p-1} \sum_{n=1}^{KM} \lambda_n^{p-1}\left(\lambda_n + p\varepsilon_n^2\right) \quad \text{for } p \ge 1 . \tag{122}$$

The cumulants for the seven special cases considered above could also be derived. However, we will only present the results for special case 6, namely uncorrelated components with proportional covariances; see (67). Again, expanding the logarithm of characteristic function (120) in a series in $i\xi$, there follows a slight simplification of (122) for the cumulants of S:

$$\chi_S(p) = (p-1)!\; 2^{p-1} \sum_{m=1}^{M} r^{(m)^p} \sum_{k=1}^{K} \lambda_k^p +$$

$$+\; p!\; 2^{p-1} \sum_{m=1}^{M} \sum_{k=1}^{K} \left(\lambda_k\, r^{(m)}\right)^{p-1} \varepsilon_k^{(m)^2} \quad \text{for } p \ge 1 . \tag{123}$$

CHARACTERISTIC FUNCTION OF PROCESSOR OUTPUT γ

With the characteristic function of sum S in hand, we can now return to the desired result (42) for the characteristic function $f_\gamma(\xi)$ of processor output γ, when conditions (38) are satisfied. Substitution of general result (92) into (42) yields

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{n=1}^{KM}\left(1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o}\lambda_n\right)\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \frac{2\tilde{E}}{N_o} \sum_{n=1}^{KM} \frac{\varepsilon_n^2}{1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o}\lambda_n\right)}\right] . \qquad (124)$$

By expanding the logarithm of (124) in a power series in $i\xi$, the cumulants $\chi_\gamma(p)$ of processor output γ are found to be given by

$$\frac{\chi_\gamma(p)}{(p-1)!\; 2^{p-1}} = \sum_{n=1}^{KM}\left(1 + \frac{2\tilde{E}}{N_o}\lambda_n\right)^p + \frac{2\tilde{E}}{N_o}\, p \sum_{n=1}^{KM}\left(1 + \frac{2\tilde{E}}{N_o}\lambda_n\right)^{p-1}\varepsilon_n^2 - K(M-2)$$

$$\text{for } p \ge 1 . \qquad (125)$$

SPECIAL CASE 1: Zero means; see (59)

Upon setting means $\{c_m(t,f)\}$ to zero in (124), the constants $\{\varepsilon_n\}$ defined in (91) become zero and (124) reduces to

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{n=1}^{KM}\left(1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o}\lambda_n\right)\right)\right]^{-\frac{1}{2}} . \qquad (126)$$

Only the eigenvalues $\{\lambda_n\}$ of KM×KM covariance matrix C of the KM random variables $\{g_m(t_k,f_k)\}$ need to be evaluated in this case.

SPECIAL CASE 2: Uncorrelated components; see (61)

The comments in the bottom paragraph of page 32 and the sequel are relevant at this point and should be reviewed. For convenience, we will utilize a normalized version of covariance matrix $C^{(m)}$ defined in (100):

$$\underline{C}^{(m)} = \frac{C^{(m)}}{R_{mm}(0,0)} = \left[\frac{R_{mm}(t_k - t_j, f_k - f_j)}{R_{mm}(0,0)}\right]_1^K . \tag{127}$$

The corresponding eigenvalue matrix $\underline{\Lambda}^{(m)}$ of normalized K×K covariance matrix $\underline{C}^{(m)}$ is then given by

$$\underline{\Lambda}^{(m)} = \frac{\Lambda^{(m)}}{R_{mm}(0,0)} \equiv \mathrm{diag}\left[\underline{\lambda}_1^{(m)} \cdots \underline{\lambda}_K^{(m)}\right] \tag{128}$$

in terms of eigenvalue matrix $\Lambda^{(m)}$ in (101), which now becomes

$$\underline{C}^{(m)} Q^{(m)} = Q^{(m)} \underline{\Lambda}^{(m)} . \tag{129}$$

Normalized modal matrix $Q^{(m)}$ is unchanged from (101). The relationship between the eigenvalues in (102) and (128) is

$$\lambda_k^{(m)} = \underline{\lambda}_k^{(m)} R_{mm}(0,0) . \tag{130}$$

When we employ special case 2 result (105) in (42), the characteristic function of processor output γ is given by

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o} \lambda_k^{(m)}\right)\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \frac{2\tilde{E}}{N_o} \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{{\varepsilon_k^{(m)}}^2}{1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o} \lambda_k^{(m)}\right)}\right] . \tag{131}$$

However, a combination of (130) and (52) yields

$$\frac{2\tilde{E}}{N_o} \lambda_k^{(m)} = \frac{2\tilde{E}}{N_o} R_{mm}(0,0) \underline{\lambda}_k^{(m)} = \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)} \text{ ,} \tag{132}$$

where we have taken note that the same signal energy $\tilde{E}$ was transmitted on all K pulses; see (38). That is, E_{1m} is the average random <u>received</u> signal energy in the m-th component of any one of the pulses. The quantity $2E_{1m}/N_o$ is a measure of the average random received "signal-to-noise ratio" in the m-th component of one pulse. At the same time, from (103),

$$\frac{2\tilde{E}}{N_o} \varepsilon_k^{(m)^2} = \left[\sum_{j=1}^{K} v_{kj}^{(m)} c_m(t_j, f_j) \left(\frac{2\tilde{E}}{N_o}\right)^{\frac{1}{2}}\right]^2 . \tag{133}$$

But since $c_m(t,f) \geq 0$ without loss of generality (see (46)), we have

$$c_m(t_j, f_j) \left(\frac{2\tilde{E}}{N_o}\right)^{\frac{1}{2}} = \left(\frac{2\tilde{E}}{N_o} c_m^2(t_j, f_j)\right)^{\frac{1}{2}} = \left(\frac{2D_{jm}}{N_o}\right)^{\frac{1}{2}} , \tag{134}$$

where D_{jm} is the <u>deterministic</u> <u>received</u> signal energy in the m-th component of the j-th pulse, as defined in (50). Then (133) can be expressed as

$$\frac{2\tilde{E}}{N_o} \varepsilon_k^{(m)^2} = \left[\sum_{j=1}^{K} v_{kj}^{(m)} \left(\frac{2D_{jm}}{N_o}\right)^{\frac{1}{2}}\right]^2 \equiv \underline{\varepsilon}_k^{(m)^2} . \tag{135}$$

The quantity $2D_{jm}/N_o$ is a measure of the received deterministic signal-to-noise ratio in the m-th component of the j-th pulse.

Upon combining these definitions, (131) is modified to

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi\left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)}\right)\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{\underline{\varepsilon}_k^{(m)2}}{1 - i2\xi\left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)}\right)} \right] . \qquad (136)$$

Here, eigenvalues $\underline{\lambda}_1^{(m)},\ldots,\underline{\lambda}_K^{(m)}$ are those of the K×K normalized covariance matrix $\underline{C}^{(m)}$ defined in (127). The constants $\underline{\varepsilon}_k^{(m)}$ are given, according to (135), by

$$\underline{\varepsilon}_k^{(m)} = \sum_{j=1}^{K} v_{kj}^{(m)} \left(\frac{2D_{jm}}{N_o}\right)^{\frac{1}{2}} . \qquad (137)$$

The cumulants of processor output γ in this case are given by

$$\frac{\chi_\gamma(p)}{(p-1)!\ 2^{p-1}} = \sum_{m=1}^{M} \sum_{k=1}^{K} \left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)}\right)^p +$$

$$+ p \sum_{m=1}^{M} \sum_{k=1}^{K} \left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)}\right)^{p-1} \underline{\varepsilon}_k^{(m)2} - K(M-2) \quad \text{for } p \ge 1 . \qquad (138)$$

SPECIAL CASE 3: Zero means and uncorrelated components

When means $\{c_m(t,f)\}$ are zero, then (50) and (137) yield $D_{km} = 0$ and $\underline{\varepsilon}_k^{(m)} = 0$, thereby causing (136) to reduce to

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi\left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)}\right)\right)\right]^{-\frac{1}{2}} . \qquad (139)$$

SPECIAL CASE 4: Uncorrelated components with identical covariances; see (64)

Now, normalized covariance matrix $\underline{C}^{(m)}$ in (127) is independent of m; in particular, we have the single K×K covariance matrix

$$\underline{C} = \frac{C}{R_{11}(0,0)} = \left[\frac{R_{11}(t_k - t_j, f_k - f_j)}{R_{11}(0,0)}\right]_1^K . \tag{140}$$

This leads to solution matrices $Q^{(m)}$ and $\underline{\Lambda}^{(m)}$ in (129) which are also independent of m; that is, (129) becomes the single K×K characteristic-value matrix equation

$$\underline{C}\, Q = Q\, \underline{\Lambda} \; ; \quad \underline{\Lambda} = \Lambda / R_{11}(0,0) = \mathrm{diag}[\underline{\lambda}_1, \ldots, \underline{\lambda}_K] . \tag{141}$$

Thus, the eigenvalues in (128) and the eigenvectors in (102) are independent of m. However, the constants $\underline{\varepsilon}_k^{(m)}$ in (137) still depend on m through their dependence on deterministic received signal energies $\{D_{jm}\}$; that is, from (137), now

$$\underline{\varepsilon}_k^{(m)} = \sum_{j=1}^{K} v_{kj} \left(\frac{2D_{jm}}{N_o}\right)^{\frac{1}{2}} . \tag{142}$$

Also, from (52), (38), and (64), we now have

$$E_{1m} = \tilde{E}_1\, R_{mm}(0,0) = \tilde{E}\, R_{11}(0,0) = E_{11} , \tag{143}$$

which is the average random received signal energy in one component of one pulse. (In this special case 4, we have the alternative result $E_{11} = E_1/M$ from (53).)

The collection of these conclusions enables us to express the characteristic function (136) of processor output γ in the closed compact form

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{k=1}^{K}\left(1 - i2\xi\left(1 + \frac{2E_{11}}{N_o}\,\underline{\lambda}_k\right)\right)\right]^{-\frac{1}{2}M} \times$$

$$\times \exp\left[i\xi \sum_{k=1}^{K} \frac{\underline{h}_k}{1 - i2\xi\left(1 + \frac{2E_{11}}{N_o}\,\underline{\lambda}_k\right)}\right] , \qquad (144)$$

where we defined the K constants

$$\underline{h}_k = \sum_{m=1}^{M} \underline{\varepsilon}_k^{(m)^2} = \sum_{m=1}^{M} \left\{\sum_{j=1}^{K} v_{kj} \left(\frac{2D_{jm}}{N_o}\right)^{\frac{1}{2}}\right\}^2 \quad \text{for } 1 \le k \le K . \qquad (145)$$

Use of (142) was made here. The cumulants of output γ are now

$$\frac{\chi_\gamma(p)}{(p-1)!\ 2^{p-1}} = M \sum_{k=1}^{K}\left(1 + \frac{2E_{11}}{N_o}\,\underline{\lambda}_k\right)^p +$$

$$+ p \sum_{k=1}^{K} \underline{h}_k \left(1 + \frac{2E_{11}}{N_o}\,\underline{\lambda}_k\right)^{p-1} - K(M-2) \quad \text{for } p \ge 1 . \qquad (146)$$

As a special subcase, suppose that the M deterministic components $\{c_m(t,f)\}$ in fading model (45) are independent of t and f; see (112). Then, from (50) and (38),

$$D_{km} = \tilde{E}_k\, c_m^2(t_k,f_k) = \tilde{E}\, c_m^2 = D_{1m} , \qquad (147)$$

where D_{1m} is the received deterministic signal energy in the m-th component of any one of the pulses. This result allows the constants in (145) to be simplified to

$$\underline{h}_k = \left(\sum_{m=1}^{M} \frac{2D_{1m}}{N_o}\right) \left(\sum_{j=1}^{K} v_{kj}\right)^2 . \tag{148}$$

Thus, in this special subcase, the constants $\{\underline{h}_k\}$ depend only on the sum of the received deterministic signal energies on all the components.

SPECIAL CASE 5: Zero means and uncorrelated components with identical covariances

Upon setting the means $\{c_m(t,f)\}$ to zero, it follows from (50), (145), and (144) that the characteristic function of processor output γ is

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{k=1}^{K}\left(1 - i2\xi\left(1 + \frac{2E_{11}}{N_o}\underline{\lambda}_k\right)\right)\right]^{-\frac{1}{2}M} . \tag{149}$$

Eigenvalues $\underline{\lambda}_1,\ldots,\underline{\lambda}_K$ correspond to K×K normalized covariance matrix $\underline{C}$ given by (140), while E_{11} is the average random received signal energy in one component of one pulse. This exact result replaces the approximation in [12; (8)].

In the special case of M = 2, that is, two components in fading model (45), the result in (149) reduces to

$$f_\gamma(\xi) = \left[\prod_{k=1}^{K}\left(1 - i2\xi\left(1 + \frac{2E_{11}}{N_o}\underline{\lambda}_k\right)\right)\right]^{-1} . \tag{150}$$

This expression agrees with [11; (24)], except for a scaling of output γ by a factor of 2. There was no need here to evaluate multiple integrals as encountered in [11; (21) - (22)].

SPECIAL CASE 6: Uncorrelated components with proportional covariances; see (67)

If we combine the general relation in (42), for the characteristic function of processor output γ, with the characteristic function in (120) for the sum S in this special case 6, we obtain the form

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o} \lambda_k \, r^{(m)}\right)\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \, \frac{2\tilde{E}}{N_o} \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{\varepsilon_k^{(m)^2}}{1 - i2\xi\left(1 + \frac{2\tilde{E}}{N_o} \lambda_k \, r^{(m)}\right)}\right] . \qquad (151)$$

Now, from (67) and (100), we find that K×K covariance matrix

$$C^{(m)} = r^{(m)} \, C \;, \quad \text{where} \quad C = \left[R_{11}(t_k - t_j, f_k - f_j)\right]_1^K , \qquad (152)$$

and $r^{(1)} = 1$. Then (101) yields

$$Q^{(m)} = Q \;, \qquad \Lambda^{(m)} = r^{(m)} \, \Lambda \;, \qquad (153)$$

where Q and Λ are the solutions to the single K×K matrix equation

$$C \, Q = Q \, \Lambda \; . \qquad (154)$$

Now, define normalized covariance matrix $\underline{C}$ as in (140), with corresponding matrix equation and eigenvalue matrix $\underline{\Lambda}$ as in (141). Then the eigenvalues are related according to

$$\lambda_k = \underline{\lambda}_k \, R_{11}(0,0) \;, \qquad (155)$$

leading to result

$$\frac{2\tilde{E}}{N_o} \lambda_k \, r^{(m)} = \frac{2\tilde{E}}{N_o} R_{11}(0,0) \, \underline{\lambda}_k \, r^{(m)} = \frac{2E_{11}}{N_o} \underline{\lambda}_k \, r^{(m)} = \frac{2E_{1m}}{N_o} \underline{\lambda}_k \, , \quad (156)$$

upon use of (38), (52), and (67). The quantity E_{1m} is the average random received signal energy in the m-th component of any one pulse. Also, from (135) and (137),

$$\frac{2\tilde{E}}{N_o} \varepsilon_k^{(m)^2} = \underline{\varepsilon}_k^{(m)^2} \, , \qquad \underline{\varepsilon}_k^{(m)} = \sum_{j=1}^{K} v_{kj} \left(\frac{2D_{jm}}{N_o}\right)^{\frac{1}{2}} \, , \quad (157)$$

since normalized modal matrix Q is independent of component number m; see (153). Use of (156) and (157) in (151) yields the characteristic function of processor output γ in the desired form

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi\left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k\right)\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{\underline{\varepsilon}_k^{(m)^2}}{1 - i2\xi\left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k\right)}\right] . \quad (158)$$

This result is a slight simplification of (136) which allowed uncorrelated components of arbitrary covariances; thus, eigenvalues $\{\underline{\lambda}_k\}$ and eigenvectors $\{V_k\}$ are independent of m here. Furthermore, (158) simplifies to the result in (144) when the covariances of the random fading components $\{g_m(t,f)\}$ in (45) are identical; see (64).

The cumulants $\{\chi_\gamma(p)\}$ of processor output γ are found by expanding the logarithm of (158) in a power series in $i\xi$:

$$\frac{X_Y(p)}{(p-1)!\ 2^{p-1}} = \sum_{m=1}^{M} \sum_{k=1}^{K} \left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k\right)^p +$$

$$+ p \sum_{m=1}^{M} \sum_{k=1}^{K} \left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k\right)^{p-1} \underline{\varepsilon}_k^{(m)^2} - K(M-2) \quad \text{for } p \geq 1 \ . \qquad (159)$$

SPECIAL CASE 7: Zero means and uncorrelated components with proportional covariances

When we set means $\{c_m(t,f)\}$ to zero, the characteristic function in (158) simplifies to

$$f_Y(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi\left(1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k\right)\right)\right]^{-\frac{1}{2}} . \quad (160)$$

For purposes of review, the quantity E_{km} is the average random received signal energy in the m-th component of the k-th pulse, while D_{km} is the received deterministic signal energy in the m-th component of the k-th pulse; see (50) and (52) as well as fading model (45) - (47). Eigenvalues $\{\underline{\lambda}_k\}$ correspond to normalized covariance matrix $\underline{C}$ in (140), while coefficients $\{v_{kj}\}$ used in (157) are the corresponding eigenvectors' components; see the K×K matrix equation in (141), which yields eigenvalue matrix $\underline{\Lambda}$ and normalized modal matrix Q.

In the sequel to (42), it was noted that the more general conditional characteristic function of γ in (27) would be treated once the fading model had been described. This latter situation with general receiver weights $\{A_k\}$ and transmitted signal energies $\{\tilde{E}_k\}$ is considered in appendix C. In particular, the unconditional characteristic function of processor output γ is derived for general correlated fading components $\{g_m(t,f)\}$ and then specialized to the case of uncorrelated fading components.

EXAMPLES OF PROBABILITY DENSITY OF POWER-SCALING q(t,f)

In an earlier section, the characteristic function of power-scaling variate $q = q(t,f)$ was derived for fading model (45), yielding general result (76). This result was then simplified for the seven special cases noted there. In this section, we will present some numerical examples of typical probability density functions of q and thereby illustrate the generality and variety of fading model (45).

The only case we will consider in this section is where the M random components $\{g_m(t,f)\}$ in (45) are uncorrelated with each other and have proportional covariances. This is special case 6; see (67). Also, for notational convenience, we consider the normalized (power) random variable

$$\phi = \frac{q}{R_{11}(0,0)} = \frac{q}{\sigma_1^2} , \tag{161}$$

where

$$\sigma_m^2 \equiv \overline{g_m^2(t,f)} = R_{mm}(0,0) = R_{11}(0,0)\; r^{(m)} \quad \text{for } 1 \le m \le M . \tag{162}$$

Here, we used (47) and (67). Then, from (86), the characteristic function of random variable ϕ is

$$f_\phi(\xi) = f_q\left(\frac{\xi}{R_{11}(0,0)}\right) = \left[\prod_{m=1}^{M}\left(1-i2\xi r^{(m)}\right)\right]^{-1/2} \exp\left[i\xi \sum_{m=1}^{M} \frac{\eta_m^2\, r^{(m)}}{1-i2\xi r^{(m)}}\right] , \tag{163}$$

where we defined (dimensionless) normalization constants

$$\eta_m = \frac{c_m}{\sigma_m} = \frac{c_m(t,f)}{R_{mm}(0,0)^{\frac{1}{2}}} \quad \text{for } 1 \le m \le M \; . \tag{164}$$

For $M = 1$, the general result in (163) simplifies to

$$f_\phi(\xi) = (1 - i2\xi)^{-\frac{1}{2}} \exp\left[\frac{i\xi\ \eta_1^2}{1 - i2\xi}\right] , \tag{165}$$

for which the corresponding PDF (probability density function) is

$$p_\phi(u) = (2\pi u)^{-\frac{1}{2}} \exp\left[-\ \frac{1}{2}\left(u + \eta_1^2\right)\right] \cosh(\eta_1\sqrt{u}) \quad \text{for } u > 0 \; . \tag{166}$$

The amplitude-scaling variate, $r_k = q_k^{\frac{1}{2}}$ from (43), has the corresponding PDF for normalized version

$$\theta = \frac{r}{\sigma_1} = \left[\frac{q}{R_{11}(0,0)}\right]^{\frac{1}{2}} = \sqrt{\phi} \; , \tag{167}$$

namely

$$p_\theta(u) = 2\ u\ p_\phi(u^2) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \exp\left[-\ \frac{1}{2}\left(u^2 + \eta_1^2\right)\right] \cosh(\eta_1 u) \ \text{for } u > 0 \; . \tag{168}$$

This PDF is displayed in figure 2 for $\eta_1 = 0(.5)3$. At the origin this PDF is finite and nonzero, with value $(2/\pi)^{\frac{1}{2}} \exp(-\eta_1^2/2)$, indicating the possibility of occasional deep fades (unless η_1 is large). As parameter η_1 gets large, this PDF approaches Gaussian; in fact, directly from (168) and figure 2, we obtain

$$p_\theta(u) \sim (2\pi)^{-\frac{1}{2}} \exp\left[-\ \frac{1}{2}\left(u - \eta_1\right)^2\right] \quad \text{for } \eta_1 > 3 \; . \tag{169}$$

The corresponding PDF for normalized power-scaling variate $\phi = q/R_{11}(0,0) = q/\sigma_1^2$ was given in (166) and is plotted in figure 3. It is seen to possess a substantial cusp at the origin, which behaves as $(2\pi u)^{-\frac{1}{2}} \exp(-\eta_1^2/2)$. This is the most severe case of deep fading that model (45) can yield, namely for M = 1.

When M = 2, that is, two components in fading model (45), the characteristic function in (163) reduces to

$$f_\phi(\xi) = \left(1 - i2\xi\right)^{-\frac{1}{2}} \left(1 - i2\xi\, r^{(2)}\right)^{-\frac{1}{2}} \exp\left[\frac{i\xi\, \eta_1^2}{1 - i2\xi} + \frac{i\xi\, \eta_2^2\, r^{(2)}}{1 - i2\xi\, r^{(2)}}\right]. \tag{170}$$

However, instead of Fourier transforming this result, the PDF of ϕ is best found by reverting to definition (45), namely $q = (c_1 + g_1)^2 + (c_2 + g_2)^2$, and performing the probability integrals directly in the g_1, g_2 plane. The result is

$$p_\phi(u) = \frac{1}{4\pi\left(r^{(2)}\right)^{\frac{1}{2}}} \int_{-\pi}^{\pi} dt\, \exp\left[-\frac{1}{2}\left(C^2 + S^2\right)\right] \quad \text{for } u > 0\ , \tag{171}$$

where auxiliary functions

$$C \equiv u^{\frac{1}{2}} \cos(t) - \eta_1\ , \quad S \equiv \left(\frac{u}{r^{(2)}}\right)^{\frac{1}{2}} \sin(t) - \eta_2\ . \tag{172}$$

As a special subcase here, for M = 2, let $r^{(2)} = 1$; that is, let both random components $\{g_m(t,f)\}$ have equal power. Then (170) yields characteristic function

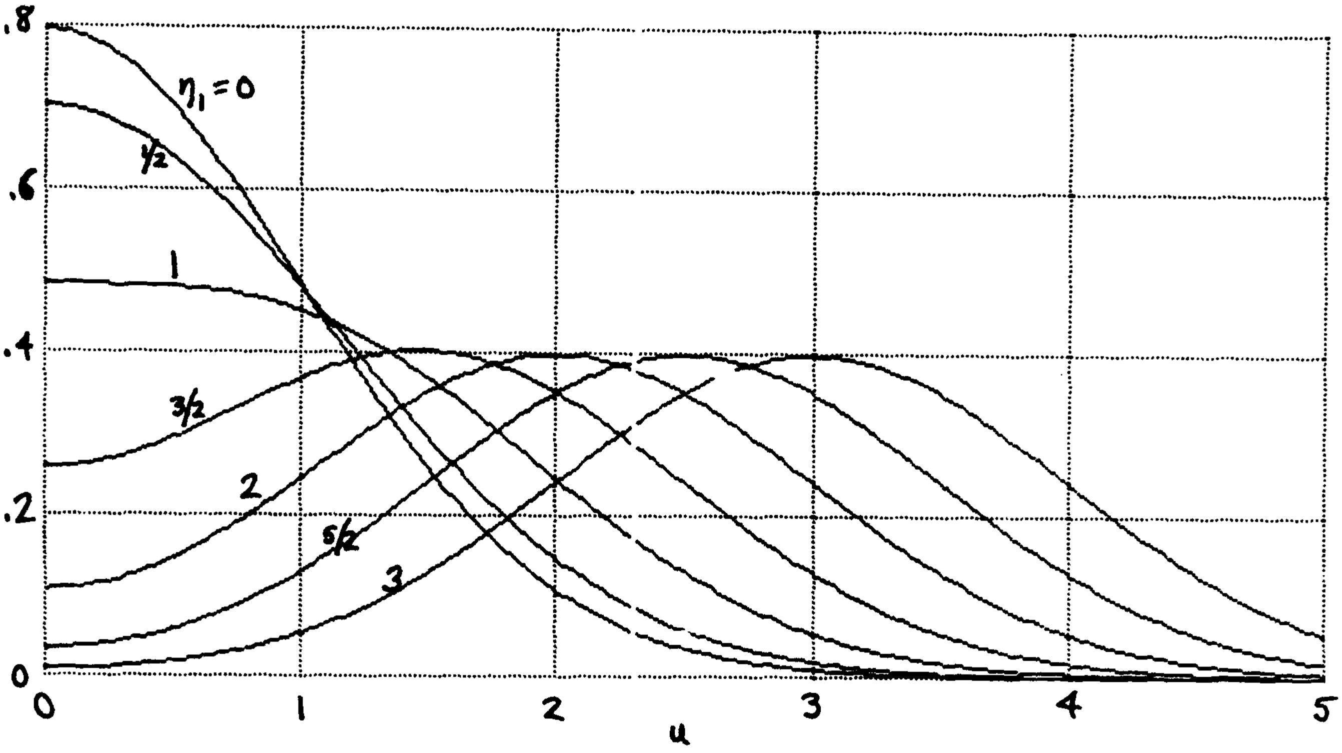

Figure 2. PDF of r/σ_1 for M = 1

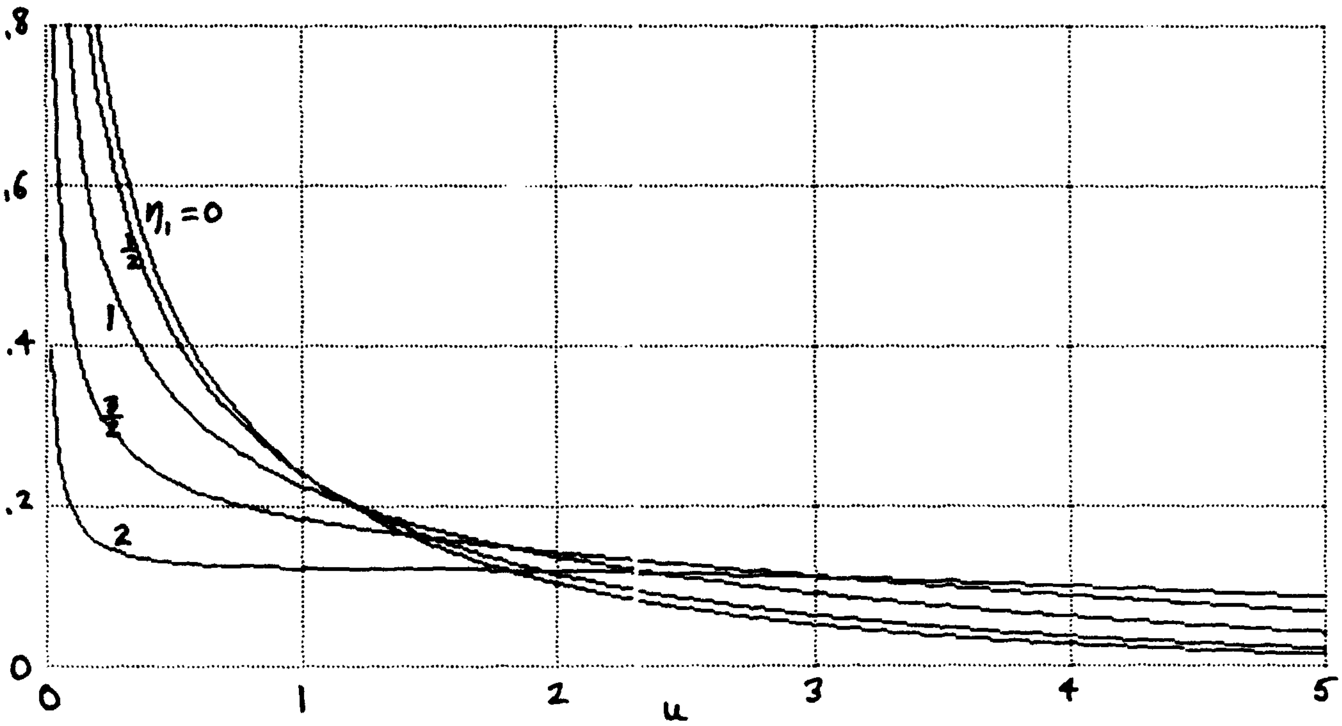

Figure 3. PDF of q/σ_1^2 for M = 1

$$f_{\phi}(\xi) = (1 - i2\xi)^{-1} \exp\left[\frac{i\xi\, \eta^2}{1 - i2\xi}\right] , \tag{173}$$

where constant

$$\eta^2 \equiv \eta_1^2 + \eta_2^2 = \frac{c_1^2 + c_2^2}{R_{11}(0,0)} \quad \text{for } M = 2 . \tag{174}$$

The corresponding probability density function for ϕ is

$$p_{\phi}(u) = \frac{1}{2}\, I_o(\eta\sqrt{u}) \exp\left[-\frac{1}{2}\left(u + \eta^2\right)\right] \quad \text{for } u > 0 . \tag{175}$$

That for $\theta = \sqrt{\phi}$ follows immediately as

$$p_{\theta}(u) = u\, I_o(\eta u) \exp\left[-\frac{1}{2}\left(u^2 + \eta^2\right)\right] \quad \text{for } u > 0 . \tag{176}$$

The PDF in (175) is displayed in figure 4. Now, the origin value

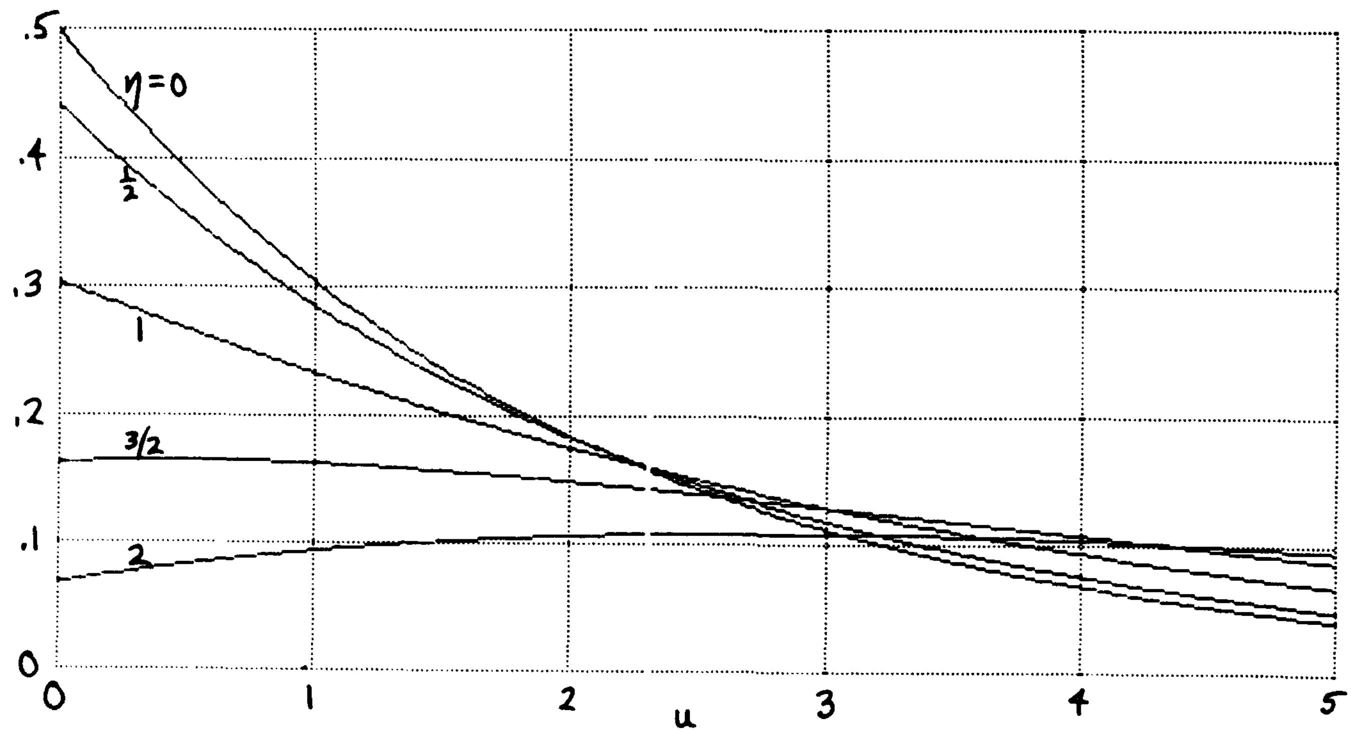

Figure 4. PDF of q/σ_1^2 for $M = 2$, $r^{(2)} = 1$

is finite, namely .5 $\exp(-\eta^2/2)$, in contrast with figure 3 for $M = 1$. The cusps that were in figure 3 are now absent for $M = 2$.

When $M = 2$ but $r^{(2)} \neq 1$, a closed form like (175) is not available for the PDF of ϕ. Instead, we must revert to the general relation in (171) - (172) and perform the numerical integration for each set of parameter values of interest. An example for $r^{(2)} = 1/2$ and $\eta_2 = \eta_1$ is given in figure 5, and a second example for $r^{(2)} = 1/2$ and $\eta_2 = \eta_1/2$ is given in figure 6. A common feature of all these densities is the finite nonzero values at the origin. That is, deep fades are still common for fading model (45) when $M = 2$.

For larger M and completely arbitrary parameters, it is necessary to numerically Fourier transform general characteristic function (163) in order to determine the PDF of ϕ. However, for the special case where

$$M \text{ is even} \quad \text{and} \quad r^{(m)} = 1 \quad \text{for } 1 \le m \le M , \tag{177}$$

then (163) reduces to

$$f_\phi(\xi) = (1 - i2\xi)^{-M/2} \exp\left[\frac{i\xi\,\eta^2}{1 - i2\xi}\right] , \tag{178}$$

where constant

$$\eta^2 = \sum_{m=1}^{M} \eta_m^2 . \tag{179}$$

The corresponding PDF of ϕ is then

$$p_\phi(u) = \frac{1}{2}\left(\frac{\sqrt{u}}{\eta}\right)^N I_N(\eta\sqrt{u}) \exp\left[-\frac{1}{2}\left(u + \eta^2\right)\right] \quad \text{for } u > 0 , \tag{180}$$

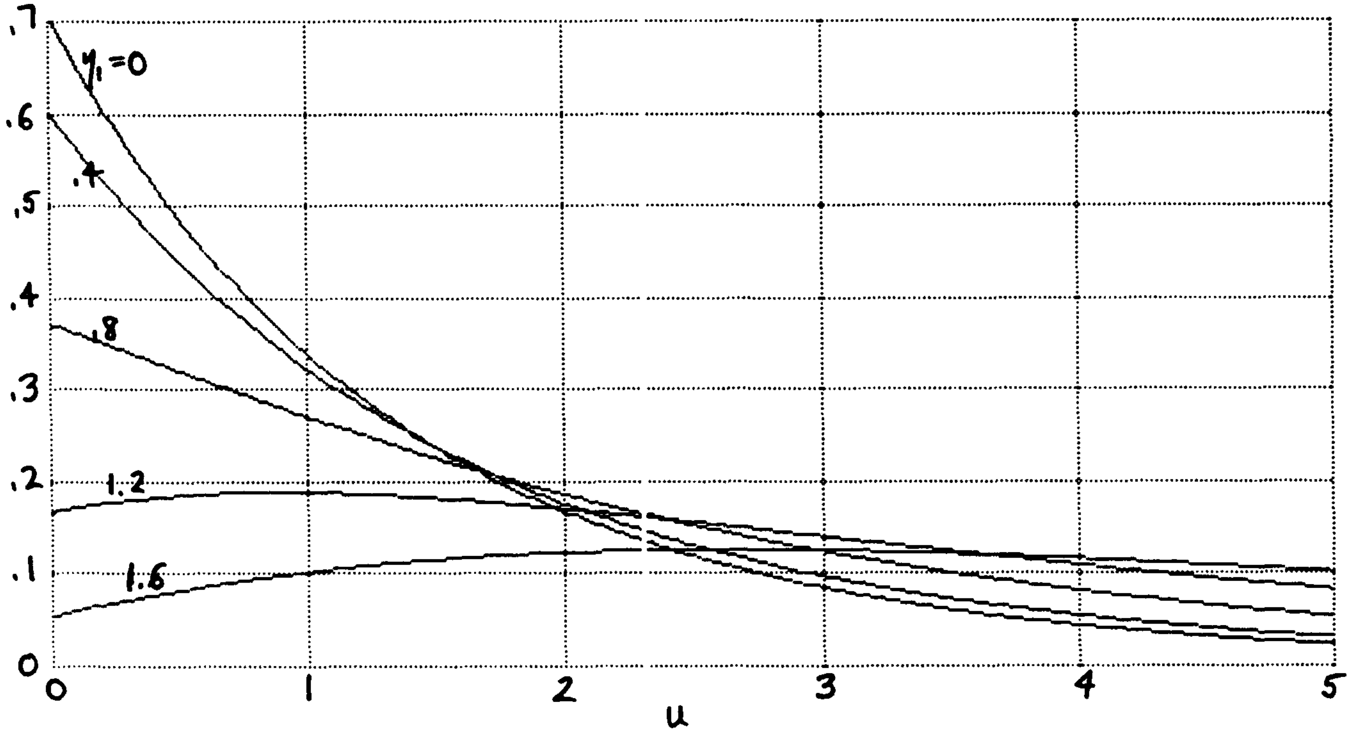

Figure 5. PDF of q/σ_1^2 for M = 2, $r^{(2)}$ = 1/2, $\eta_2 = \eta_1$

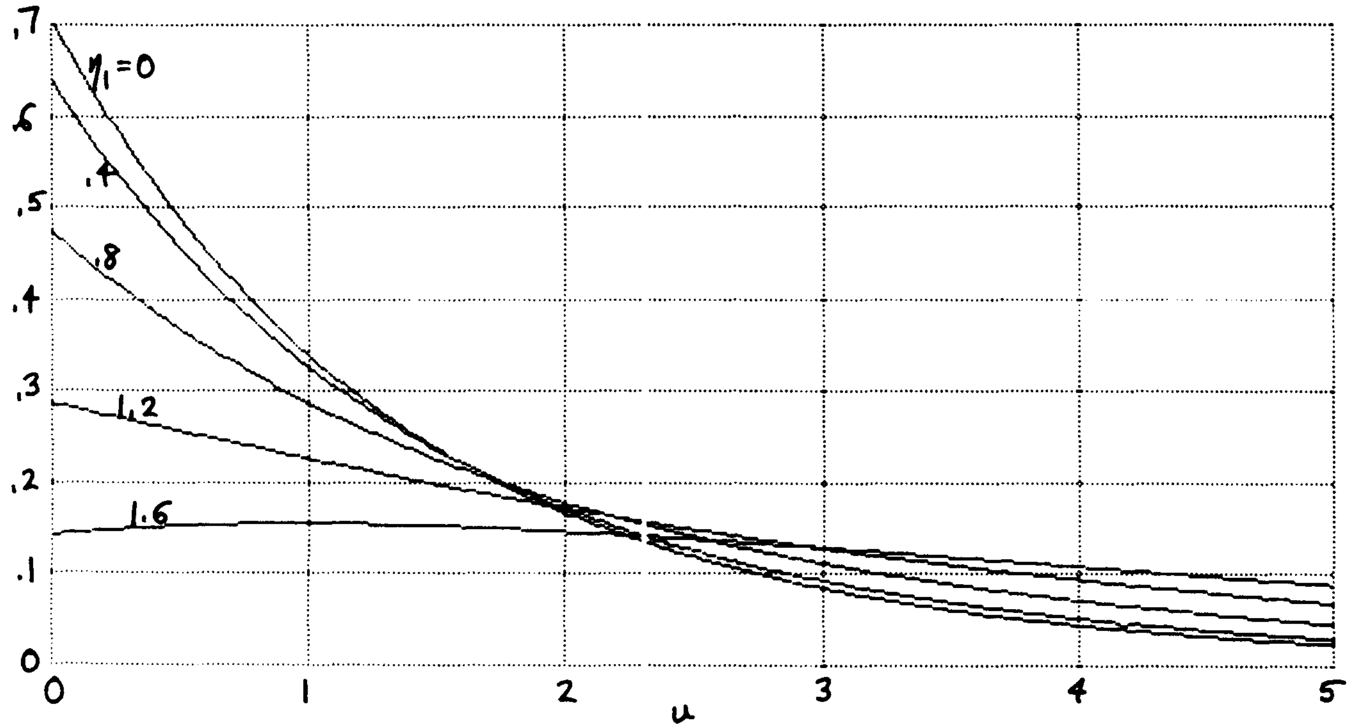

Figure 6. PDF of q/σ_1^2 for M = 2, $r^{(2)}$ = 1/2, $\eta_2 = \eta_1/2$

where $N = M/2 - 1$. This reduces to (175) for $M = 2$.

The PDF for $\theta = \sqrt{\phi}$ follows as given by the rule given in (168), namely

$$p_\theta(u) = u \left(\frac{u}{\eta}\right)^N I_N(\eta u) \exp\left[-\frac{1}{2}\left(u^2 + \eta^2\right)\right] \quad \text{for } u > 0 . \tag{181}$$

This reduces to (176) for $M = 2$.

Two examples of PDF (180) for random variable ϕ are displayed in figure 7 for $M = 4$ and in figure 8 for $M = 6$, respectively. The former PDF is zero at the origin, while the latter is zero and has zero first derivative at the origin. Thus, deep fades are less likely for the larger values of M in fading model (45).

A general procedure for determining the probability density function $p_\phi(u)$ of random variable ϕ from characteristic function $f_\phi(\xi)$ in (163), for arbitrary M, $\{r^{(m)}\}$, $\{\eta_m\}$, is presented at the end of appendix D. It employs some efficient recursions for fast and accurate numerical evaluation of $p_\phi(u)$.

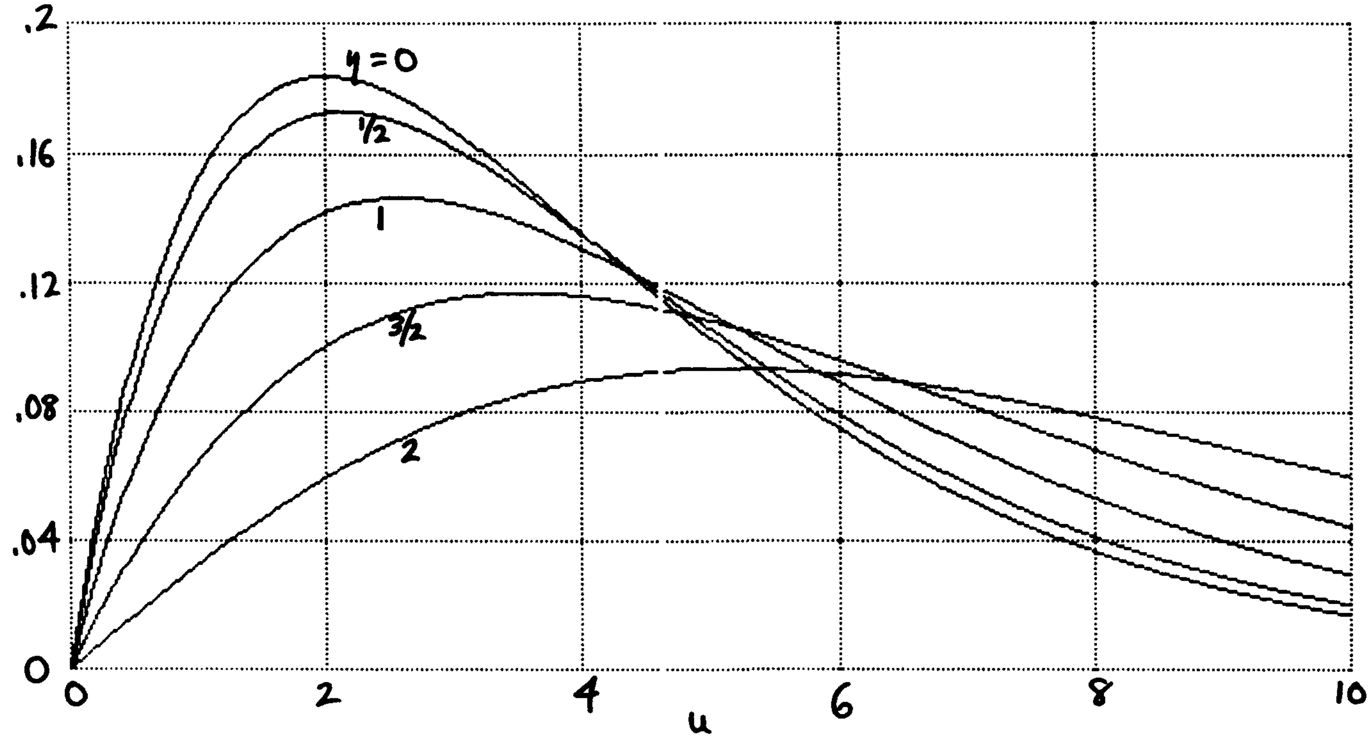

Figure 7. PDF of q/σ_1^2 for M = 4, $r^{(m)}$ = 1 for all m

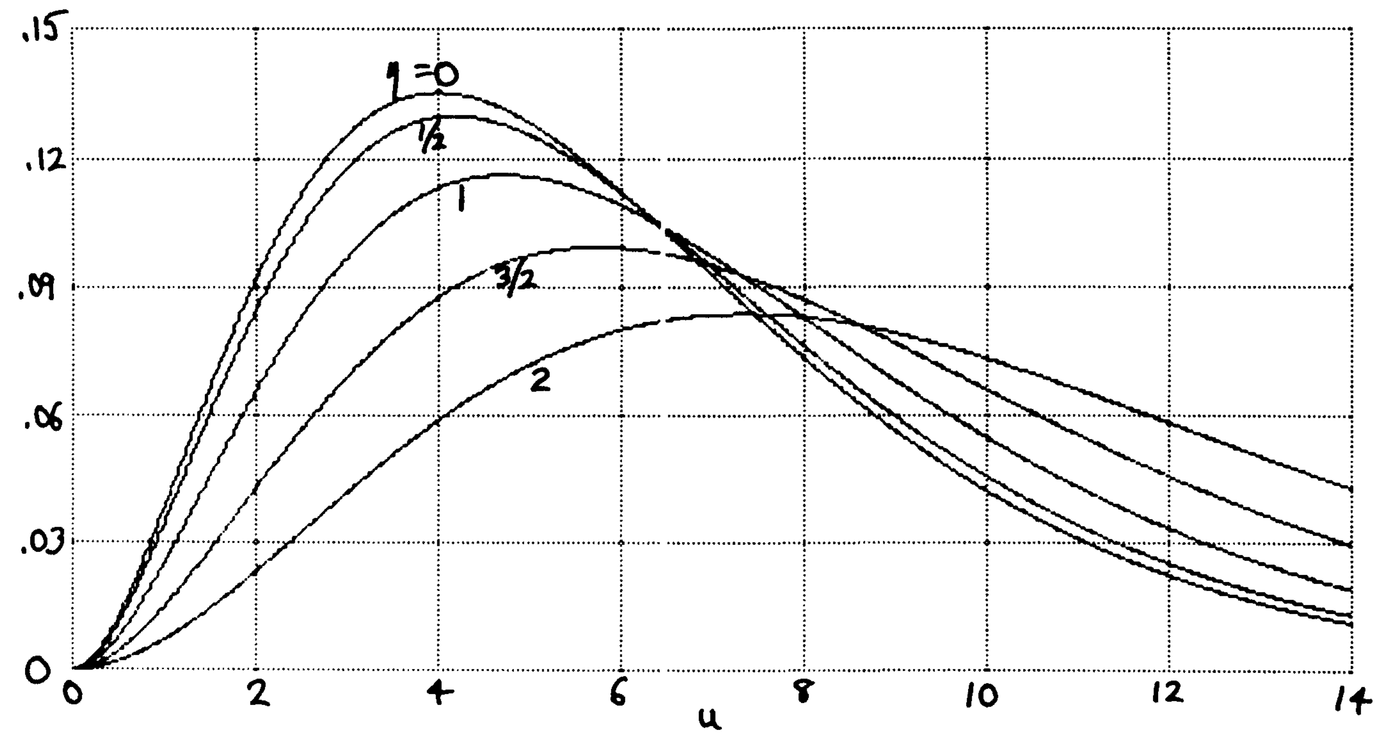

Figure 8. PDF of q/σ_1^2 for M = 6, $r^{(m)}$ = 1 for all m

PROBABILITY DISTRIBUTION OF PROCESSOR OUTPUT γ

The characteristic functions $f_\gamma(\xi)$ of processor output γ, under a variety of special cases, were derived in an earlier section; in particular, see (124) - (160). Although these are compact closed forms, they can encounter computational problems if we attempt to directly utilize Fourier transforms to determine the corresponding probability density functions and exceedance distribution functions. In particular, since the characteristic functions decay only as ξ^{-K} as $\xi \to \infty$, truncation error can become a significant problem, especially for small K, the number of signal pulses.

EXPANSION FOR DISTRIBUTION FUNCTION

In appendix D, a series expansion for $f_\gamma(\xi)$ and the corresponding probability density function $p_\gamma(u)$ are derived in a form which involves <u>only</u> positive expansion coefficients and terms. Furthermore, efficient and accurate recursions are developed for the evaluation of the coefficients, the probability density function, and the exceedance distribution function.

As an example of the general procedure, the results for general characteristic functions (136) and (158) are summarized here. Define, for $1 \le m \le M$, $1 \le k \le K$,

$$T_{mk} = \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)} \quad \text{or} \quad T_{mk} = \frac{2E_{1m}}{N_o} \underline{\lambda}_k \; , \qquad e_{mk} = \underline{\varepsilon}_k^{(m)^2} \; . \tag{182}$$

Then, the exceedance distribution function of processor output γ is given by (D-16) as

$$\text{Prob}(\gamma > u) = F \sum_{p=0}^{\infty} g_p \, H_{K+p-1}\left(\frac{u}{2}\right) \quad \text{for } u > 0 , \tag{183}$$

where the quantities required are found in the following fashion.

$$F = \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 + T_{mk}\right)\right]^{-\frac{1}{2}} , \tag{184}$$

$$\alpha_o = -\frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{e_{mk}}{1 + T_{mk}} ,$$

$$\alpha_p = \frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{T_{mk}^{p-1}}{(1 + T_{mk})^p} \left[\frac{T_{mk}}{p} + \frac{e_{mk}}{1 + T_{mk}}\right] \quad \text{for } p \geq 1 , \tag{185}$$

$$g_o = \exp(\alpha_o) , \quad g_p = \frac{1}{p} \sum_{n=1}^{p} n \, \alpha_n \, g_{p-n} \quad \text{for } p \geq 1 , \tag{186}$$

$$H_n(x) = \exp(-x) \sum_{k=0}^{n} \frac{x^k}{k!} \quad \text{for } n \geq 0 , \; x \geq 0 . \tag{187}$$

Since the eigenvalues of a covariance matrix can never be negative, it is seen from (185) that all the $\{\alpha_p\}$ are nonnegative, with the exception of α_o. However, α_o is used only once in (186) to generate a positive g_o, while the recursion in (186) for $\{g_p\}$ utilizes only nonnegative quantities. Thus, all the coefficients $\{g_p\}$ and scale factor F in (183) are nonnegative. The function $H_n(x)$ in (187) is obviously positive

for $x \ge 0$, thereby guaranteeing that no negative terms will appear in expansion (183). Furthermore, there is a very efficient recursion for the $\{H_n(x)\}$; see (D-18). A sample program for the evaluation of a modified form of (183) is given in appendix D.

In the absence of signal, all the $\{T_{mk}\}$ and $\{e_{mk}\}$ in (182) are zero. Then, all the $\{\alpha_p\}$ and $\{g_p\}$ become zero except for g_o which is 1. Expansion (183) for the exceedance distribution function of processor output γ then reduces to simply the one term $H_{K-1}(u/2)$, which is consistent with the noise-only characteristic function $(1 - i2\xi)^{-K}$ for processor output γ; see (42), for example. That is, the false alarm probability is

$$P_F = H_{K-1}\left(\frac{u}{2}\right) . \qquad (188)$$

An error bound for sum (183), terminated at the $p = N$ term, is given in (D-20). Also, the modifications required to treat the slightly different forms of characteristic functions encountered in (124) and (144) are presented in (D-23) - (D-26). Finally, a refinement of the expansion procedure in (183), which is more rapidly convergent and useful for larger signal-to-noise ratios, is given in (D-27) - (D-36).

FUNDAMENTAL INPUTS AND COMPUTATIONAL PROCEDURE

In the remainder of this section, we will restrict attention to characteristic function (158) which pertains to special case 6, namely uncorrelated components with proportional covariances; see (67). However, before we list the fundamental inputs that are required to conduct the numerical evaluation of the exceedance distribution function of processor output γ in this case, we make an additional modification for later convenience in plotting and comparison.

From (52), (38), and (67), we have

$$E_{km} = E_{1m} = \tilde{E}\; R_{11}(0,0)\; r^{(m)} \; . \tag{189}$$

Then, using (53), the average random received signal energy in one pulse is

$$E_1 = \sum_{m=1}^{M} E_{1m} = \tilde{E}\; R_{11}(0,0) \sum_{m=1}^{M} r^{(m)} \; . \tag{190}$$

This enables us to express the signal-to-noise ratio measure as

$$\frac{E_{1m}}{N_o} = \frac{E_1}{N_o}\,\psi_m \; , \quad \text{where} \quad \psi_m \equiv \frac{r^{(m)}}{\sum_{m=1}^{M} r^{(m)}} \quad \text{for } 1 \le m \le M \; . \tag{191}$$

The quantities $\{\psi_m\}$ represent the fractional strengths of each of the M random components $\{g_m(t,f)\}$ in fading model (45).

The fundamental inputs required to evaluate the exceedance distribution function of processor output γ are now

K, number of signal pulses in figure 1,

M, number of fading components in (45),

E_1/N_o, signal energy to noise density ratio,

$\{r^{(m)}\}$ for $1 \le m \le M$; see (67),

$\{D_{km}/N_o\}$ for $1 \le k \le K$, $1 \le m \le M$; see (50),

$R_{11}(\tau,\nu)/R_{11}(0,0)$; see (47),

$\{t_k\}$ and $\{f_k\}$ for $1 \le k \le K$; see (44). (192)

The quantity E_1/N_o is a measure of the receiver input average random signal-to-noise ratio for one signal pulse, while $r^{(m)}$ is the relative strength of the m-th fading component. Ratio D_{km}/N_o is a measure of the receiver input deterministic signal-to-noise ratio for the k-th signal pulse and m-th component. The function $R_{11}(\tau,\nu)/R_{11}(0,0)$ is the normalized fading covariance function for the medium at time separation τ and frequency separation ν. Parameters t_k and f_k are the time and frequency locations, respectively, of the k-th signal pulse in time,frequency space.

The first quantity that must be computed is the normalized K×K covariance matrix $\underline{C}$ given by (140). Then, its eigenvalue matrix $\underline{\Lambda}$ and normalized modal matrix Q are found by solving characteristic-value matrix equation (141). This yields eigenvalues $\{\underline{\lambda}_k\}$ and eigenvectors $\{V_k\}$ for $1 \le k \le K$. The components of vector V_k are $v_{k1}, \cdots, v_{kK}$; see (102). We then compute $\underline{\varepsilon}_k^{(m)}$ from (142) for $1 \le k \le K$, $1 \le m \le M$. At this point, the parameters $\{T_{mk}\}$ and $\{e_{mk}\}$ in (182) can be computed for $1 \le k \le K$, $1 \le m \le M$, and the procedure in (183) - (187) can be employed. In particular, (182) and (191) yield

$$T_{mk} = \frac{2E_1}{N_o} \psi_m \underline{\lambda}_k \quad \text{for} \quad 1 \le m \le M \, , \quad 1 \le k \le K \, . \tag{193}$$

NUMERICAL PERFORMANCE RESULTS

The first set of examples are selected to enable a comparison with the approximate procedure and results in [12]. In figure 9, below, we plot the required per-pulse input SNR (signal-to-noise ratio) measure E_1/N_o (in dB) versus the number of signal pulses K, for values of the adjacent-pulse normalized (amplitude) covariance, Cov, equal to 0, .5, $\sqrt{.5}$, and 1. In particular, the covariance $R_{11}(\tau,\nu)$ was taken as exponential in τ, and the signal pulses have no frequency shifts $\{f_k\}$ and are equally spaced in time locations $\{t_k\}$. Also, the KM deterministic signal-to-noise ratios $\{D_{km}/N_o\}$ are all zero, and the M strengths $\{r^{(m)}\}$ are all equal to 1; this duplicates the situation in [12]. The particular case in figure 9 pertains to false alarm probability P_F = 1E-6, detection probability P_D = .5, and M = 1 fading component; this last choice corresponds to parameter m = 1/2 in [12]. The results here in figure 9 for Cov = 0 and Cov = 1 agree precisely with ρ = 0 and ρ = 1 in [12; figure 11], as expected, while the results for Cov = $\sqrt{.5}$ are in rather good agreement with those for ρ = .5. This selection of parameters reflects the property that $\rho = \text{Cov}^2$; see the section entitled Program in appendix D for additional details.

The only change in figure 10 is to require a larger detection probability, namely P_D = .9. Although the Cov = 0 and 1 results

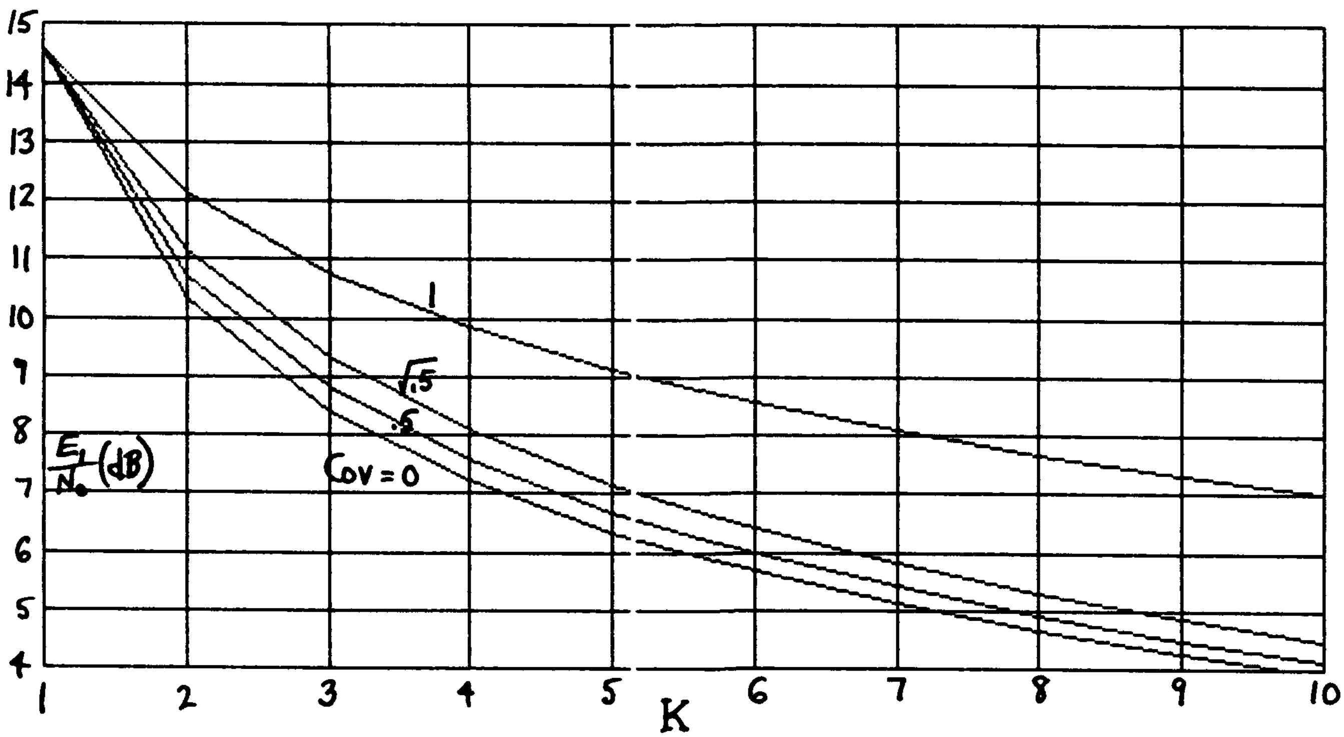

Figure 9. Required SNR for P_F = 1E-6, P_D = .5, M = 1

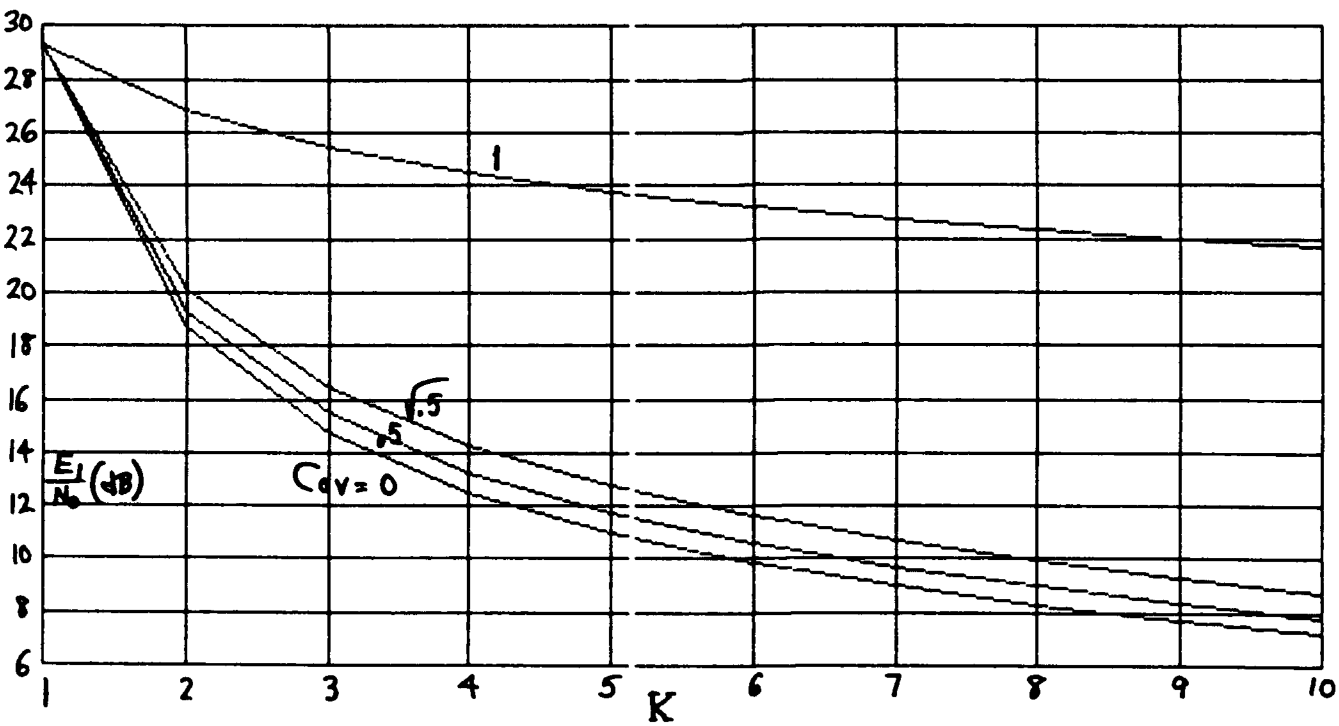

Figure 10. Required SNR for P_F = 1E-6, P_D = .9, M = 1

agree with [12; figure 12], the exact results here for Cov = $\sqrt{.5}$ are markedly different from the approximate results for ρ = .5 in the earlier work. For example, 2.6 dB less is required for K = 2 and K = 3, while 2.3 dB less is required for K = 4. That is, the earlier approximation in [12] was somewhat pessimistic in its performance predictions.

We must observe from figure 10 that the effect of correlated fading is not overly significant until the normalized covariance approaches 1. For example, for K = 2, the cost of Cov increasing from 0 to $\sqrt{.5}$ is 1.4 dB, while the cost of Cov going from $\sqrt{.5}$ to 1 is an additional 6.7 dB. The distinction is even greater for K = 10, with the same comparison requiring 1.6 dB versus an additional 12.9 dB.

The example in figure 11 pertains to M = 2 fading components, (which corresponds to m = 1 in [12]); all other parameters are the same as figure 9 above. These results can be compared directly with [12; figure 7]; they reveal identical performance for Cov = 0 and 1, and rather good agreement for Cov = $\sqrt{.5}$ (versus ρ = .5).

When the detection probability P_D is increased to .9 in figure 12, these exact results reveal that lower values of E_1/N_o are required than the approximation in [12; figure 8] predicted, at least for Cov = $\sqrt{.5}$ (ρ = .5). Also, as was seen in figure 10, the cost of normalized covariance Cov approaching 1 is very significant in terms of increased signal level; that is, the Cov = $\sqrt{.5}$ curves are well below the Cov = 1 curves in figures 10

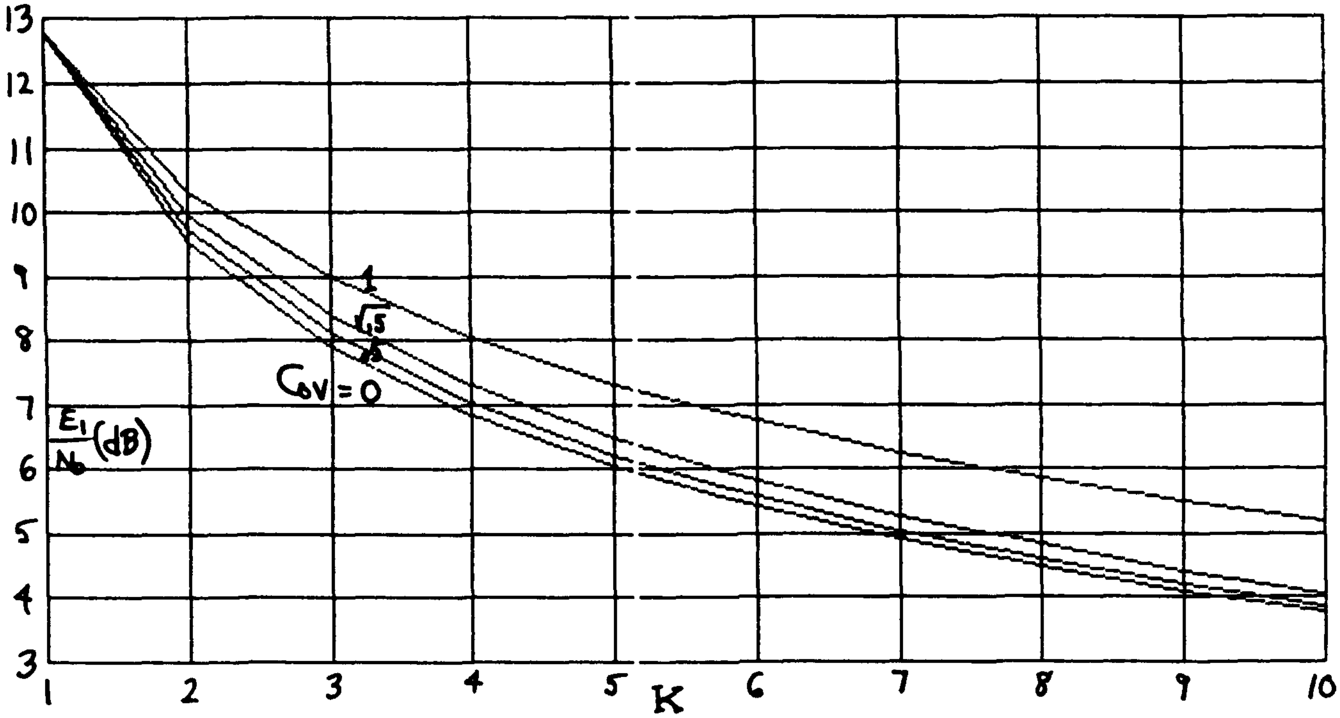

Figure 11. Required SNR for P_F = 1E-6, P_D = .5, M = 2

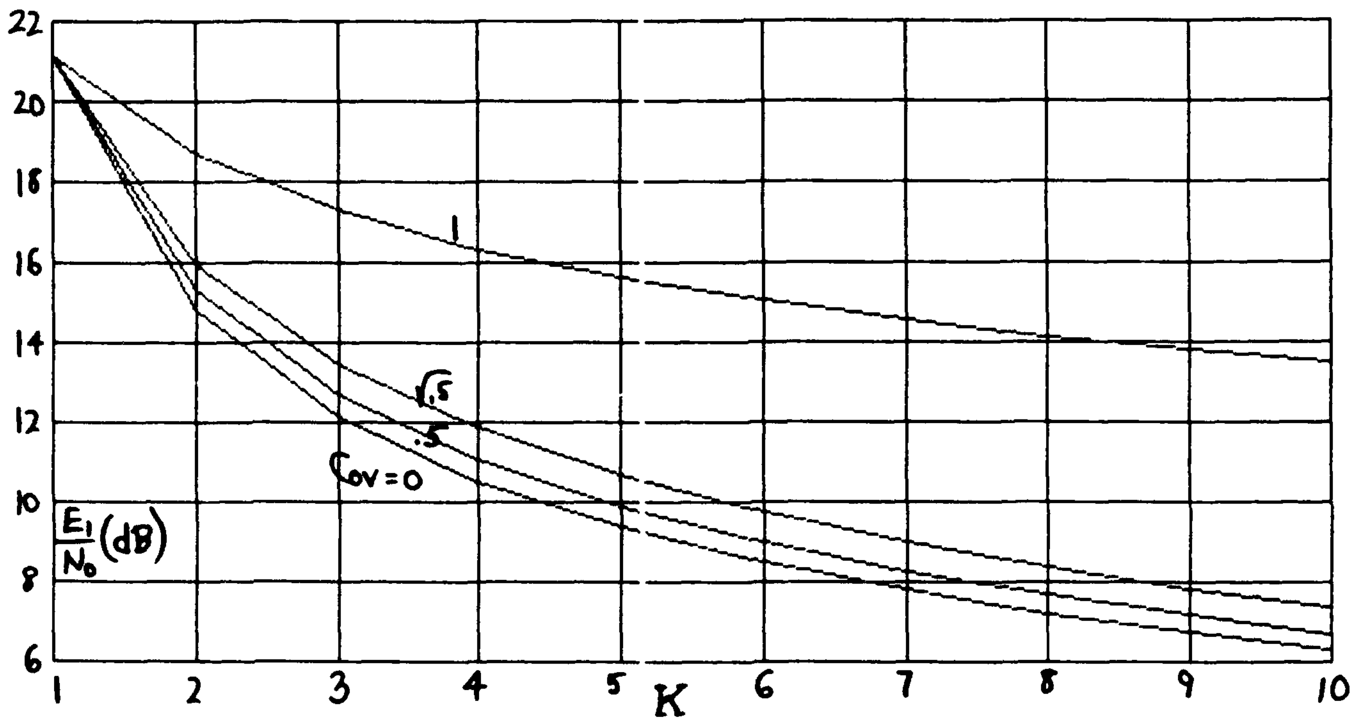

Figure 12. Required SNR for P_F = 1E-6, P_D = .9, M = 2

and 12, especially for the larger K values.

The detrimental effect of highly correlated fading pulses is studied quantitatively in figure 13, where normalized covariance Cov is varied from 0 to 1 as the other various parameters are kept fixed. The number K of signal pulses is kept at 2. These plots reveal that for detection probability $P_D = .5$, increased Cov does not lead to a significantly large increase in required input signal level. However, for the better detection probability of $P_D = .9$, higher covariances can be very damaging, requiring additional signal strength to maintain the desired level of performance. For example, as Cov increases from .9 to 1 in the uppermost example in figure 13, the signal must be increased by 4.9 dB. (The kinks in the curves are due to discretization of the abscissa at increment .1 for Cov.)

When K is increased to 6, the results in figure 14 reveal this effect in a more pronounced fashion. An additional 9 dB is now required when Cov is increased from .9 to 1 for the upper curve.

All of the above results have had deterministic signal-to-noise ratio measures $\{D_{km}/N_o\}$ equal to zero. The program listed in appendix D has the capability of incorporating arbitrary values for these parameters as well as others, such as $\{r^{(m)}\}$. An example where all the potential is exercised, and all parameters have nonzero values, is displayed in figure 15. Here, the detection probability P_D is varied from .5 to .999 and the required E_1/N_o is calculated (in dB). It is seen that a sharp

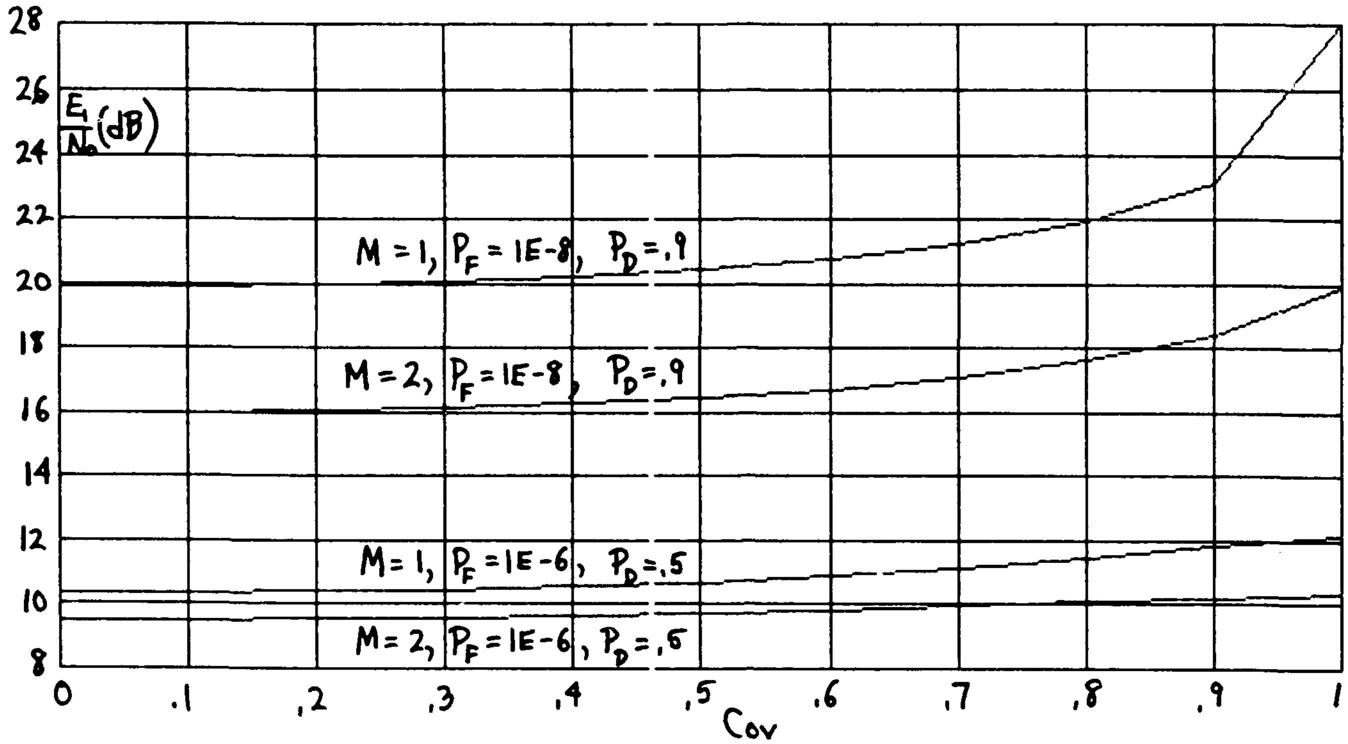

Figure 13. Required SNR versus Cov, K = 2

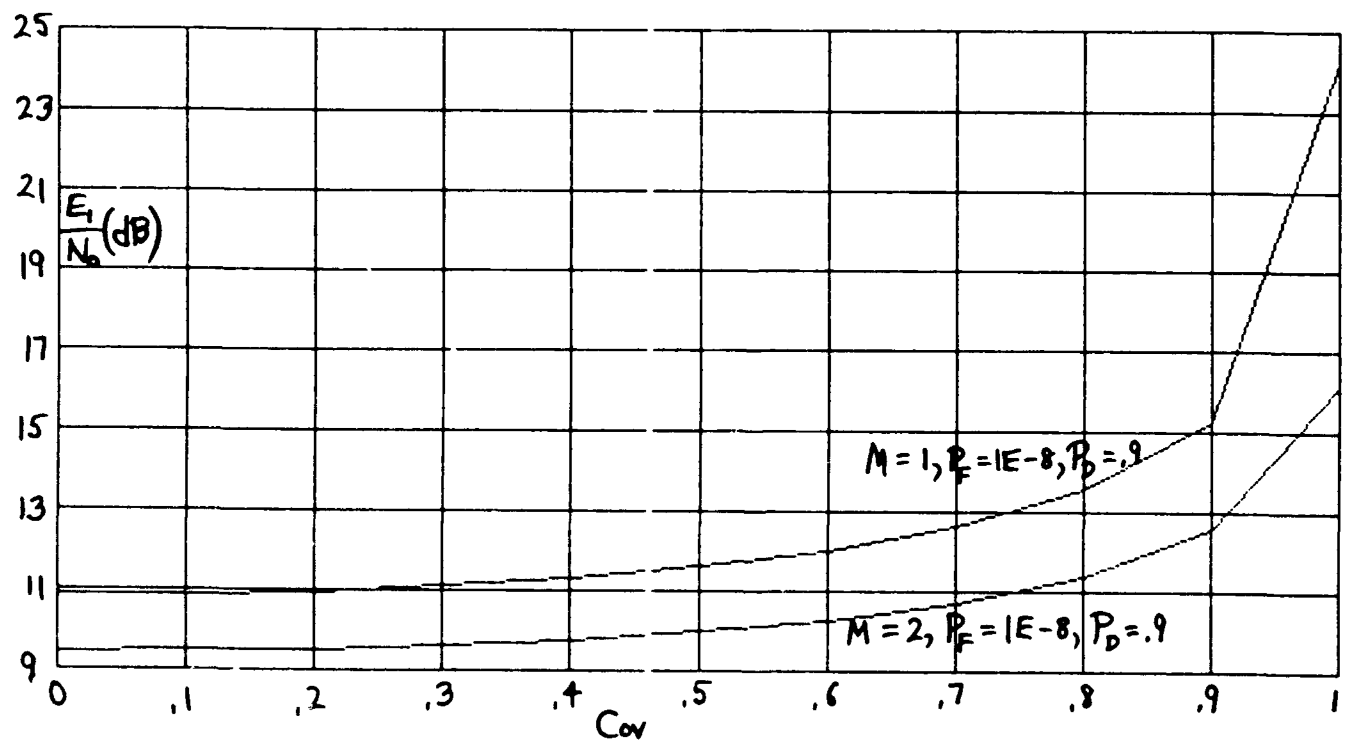

Figure 14. Required SNR versus Cov, K = 6

increase is observed as P_D increases above .9, eventually tending to ∞ as $P_D \to 1$. The particular parameter values are listed below:

$$P_F = 1E\text{-}6, \quad K = 4, \quad M = 2,$$
$$r^{(m)} = 1 \quad \text{for } 1 \le m \le M,$$
$$D_{km}/N_o = 1 \text{ for } 1 \le k \le K,\ 1 \le m \le M,$$
$$\{t_k\} = 1,2,3,4, \qquad \{f_k\} = 1,4,2,3,$$
$$\text{Cov} = \exp\left(-\frac{\tau^2}{11} - \frac{\nu^2}{10}\right) . \tag{194}$$

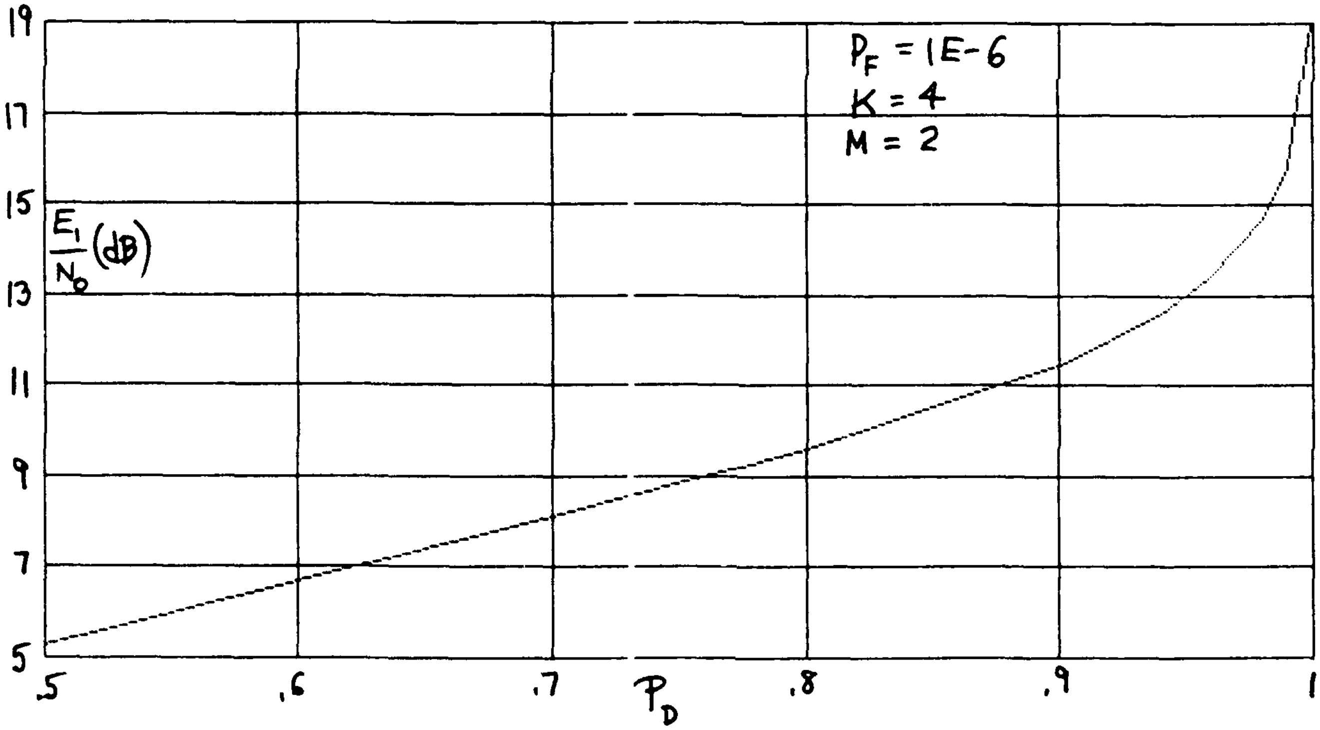

Figure 15. Required SNR versus P_D

SUMMARY

The characteristic function of processor output γ in figure 1 has been derived in closed form for a wide variety of fading conditions and signal formats. The corresponding probability density functions and exceedance distribution functions of γ have then been expanded in convergent series, by means of a novel expansion technique which make efficient use of recursions for rapid and accurate evaluation. These series typically require on the order of 30 to 60 terms for ten decimal accuracy. A program is given which incorporates all these features, including allowance for both deterministic and random components of arbitrary strengths in the fading medium.

One of the most useful results pointed out by this study is that the degree of correlation between the fading signal pulses can be fairly significant without suffering great degradations. That is, when the normalized covariance approaches 1, meaning that all signal pulses tend to fade together, the performance losses are potentially large; however, for coefficients below .5, the losses are not too significant.

Comparison of these exact results with an earlier approximate procedure [12] reveals that the earlier approach generally gives pessimistic predictions of performance when the normalized covariance of the fading is intermediate between 0 and 1. In some cases, the discrepancy can be several dB. This result indicates and emphasizes the need for accurate treatment of systems which must perform well, that is, yield high detection

probabilities in fading media. An extension of this work to a fading medium in which the background noise level is unknown and must be estimated from a finite sample in a noise-only region of time,frequency space, is currently underway by this author and will be reported on shortly.

The major result utilized here is the characteristic function of a quadratic form in correlated nonzero-mean Gaussian random variables; this result is presented in appendix B. It relies on the ability to solve the generalized eigenvalue problem; this latter procedure and solution is presented in appendix E. A related problem involving a slightly more general bilinear form is treated in appendix F. Finally, the characteristic function of the most general <u>complex</u> form with both first-order and second-order terms is solved in appendix G. These results are not utilized in this technical report but are presented for completeness and for possible future use and reference.

APPENDIX A. ANALYTIC AND COMPLEX ENVELOPE PROCESSES

Let $n(t)$ be a stationary zero-mean real random process with covariance $R_n(\tau)$ and double-sided spectrum $G_n(f)$:

$$R_n(\tau) = \overline{n(t)\ n(t-\tau)}\ , \qquad G_n(f) = \int d\tau\ \exp(-i2\pi f\tau)\ R_n(\tau)\ . \tag{A-1}$$

The analytic process $n_+(t)$ is generated by eliminating the negative frequencies in $n(t)$ and by doubling its positive frequency contributions; that is, $n(t)$ is passed through a filter with transfer function $2\ U(f)$, where U is the unit step function.

$$\xrightarrow{\ n(t)\ } \boxed{2\ U(f)} \xrightarrow{\ n_+(t)\ }$$

Figure A-1. Generation of Analytic Process

The analytic process can be expressed in terms of its real and imaginary parts according to

$$n_+(t) = n(t) + i\ n_H(t)\ , \tag{A-2}$$

where $n_H(t)$ is the Hilbert transform of $n(t)$. The complex envelope process $\underline{n}(t)$ is obtained by frequency down-shifting analytic process $n_+(t)$, in order to center its one-sided spectrum about $f = 0$. Thus, if f_o is a representative center frequency of $n_+(t)$, then

$$\underline{n}(t) = n_+(t)\ \exp(-i2\pi f_o t)\ . \tag{A-3}$$

The original real process n(t) then follows from (A-2) and (A-3):

$$n(t) = \mathrm{Re}\Big(n_+(t)\Big) = \mathrm{Re}\Big(\underline{n}(t)\ \exp(i2\pi f_o t)\Big)\ . \tag{A-4}$$

The spectrum of $n_+(t)$ follows from figure A-1 as

$$G_{n_+}(f) = 4\ U(f)\ G_n(f)\ , \tag{A-5}$$

while the spectrum of $\underline{n}(t)$ is obtained from (A-3) according to

$$G_{\underline{n}}(f) = G_{n_+}(f+f_o) = 4\ U(f+f_o)\ G_n(f+f_o)\ . \tag{A-6}$$

The covariances corresponding to (A-5) and (A-6) are

$$\overline{n_+(t)\ n_+^*(t-\tau)} \quad \text{and} \quad \overline{\underline{n}(t)\ \underline{n}^*(t-\tau)}\ , \tag{A-7}$$

respectively. The two complementary covariances are zero; that is,

$$\overline{n_+(t)\ n_+(t-\tau)} = 0\ , \quad \overline{\underline{n}(t)\ \underline{n}(t-\tau)} = 0\ . \tag{A-8}$$

These results follow from the fact that transfer function 2 U(f) in figure A-1 is single-sided; that is, U(f) = 0 for f < 0.

A situation that frequently arises in practice is where n(t) is a noise process with a spectrum $G_n(f)$ that is flat in a broad band of width W in the neighborhood of f_o, which essentially covers the signals and filters of interest. The spectra of the various processes are illustrated in figure A-2, where N_d is the double-sided noise spectral density level of n(t) in the neighborhood of $\pm f_o$. The spectrum $G_{\underline{n}}(f)$ of the complex envelope $\underline{n}(t)$ is flat in the neighborhood of f = 0 and has level 4 N_d watts/Hz. Therefore, its covariance

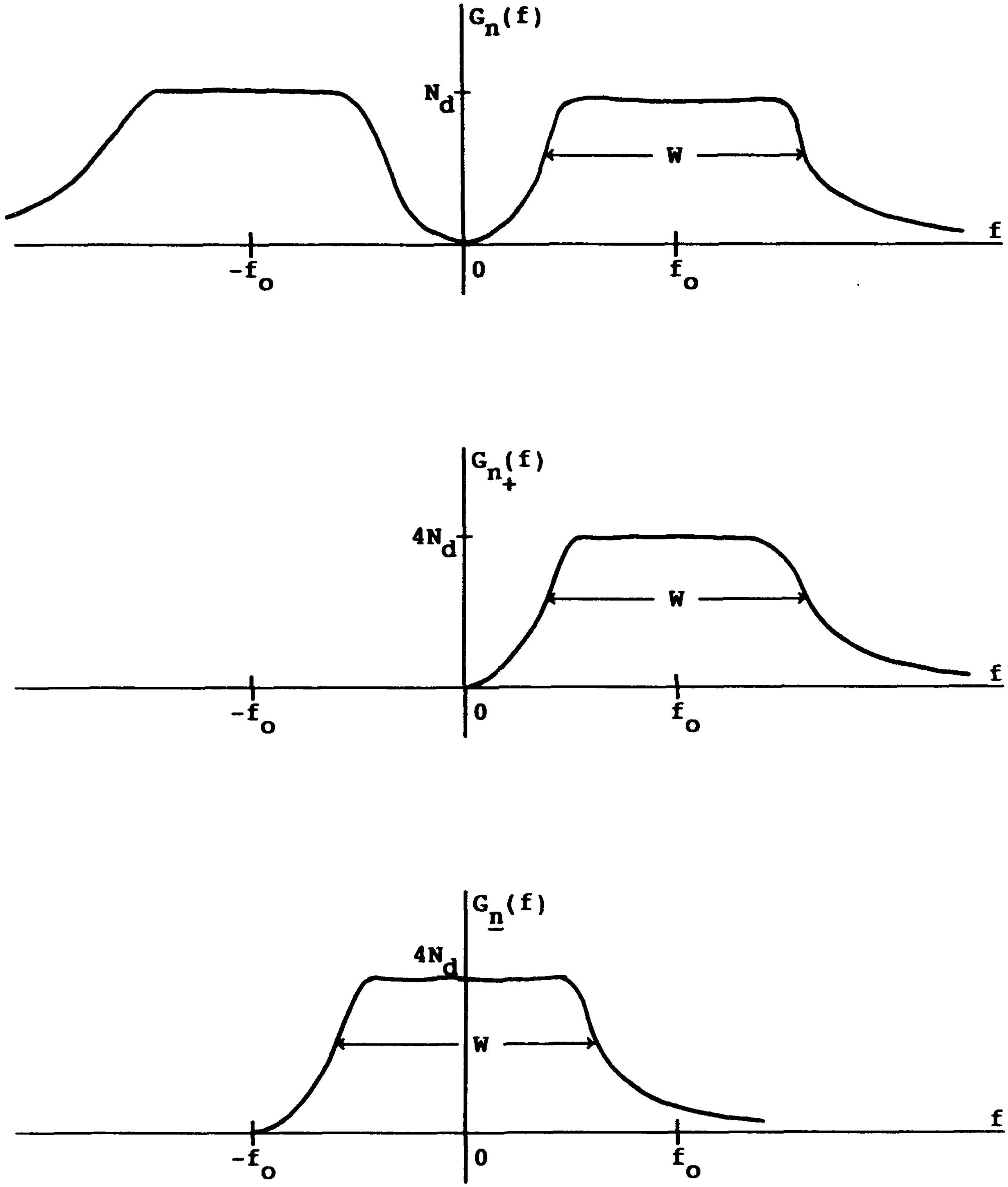

Figure A-2. Spectra of Analytic and Complex Envelope Processes

$$R_{\underline{n}}(\tau) = \int df \exp(i2\pi f\tau)\, G_{\underline{n}}(f) \tag{A-9}$$

is (relative to the signal and filter time functions) a sharp pulse centered at $\tau = 0$, with area $G_{\underline{n}}(0) = 4\,N_d$. Then, a good approximation for many purposes is to say that

$$R_{\underline{n}}(\tau) \cong 4\,N_d\,\delta(\tau)\ . \tag{A-10}$$

There is no need to let W and f_o tend to infinity in order to utilize this result; it only requires that W be somewhat larger than the bandwidths of the signals and filters of interest.

In terms of the one-sided noise spectral density level of n(t), that is, $N_o = 2\,N_d$, approximation (A-10) becomes

$$R_{\underline{n}}(\tau) \cong 2\,N_o\,\delta(\tau)\ . \tag{A-11}$$

This result may be compared with [19; page 48, (3.10) - (3.11) and page 72, (6.21) - (6.22)], where his N is our N_o.

It should be noted that original covariance $R_n(\tau)$ in (A-1) <u>cannot</u> be recovered from this approximation. That is, via (A-4),

$$R_n(\tau) = \frac{1}{2}\,\mathrm{Re}\Big(R_{\underline{n}}(\tau)\,\exp(i2\pi f_o\tau)\Big) = \tag{A-12}$$

$$\overset{?}{\cong} \frac{1}{2}\,\mathrm{Re}\Big(4\,N_d\,\delta(\tau)\,\exp(i2\pi f_o\tau)\Big) = 2\,N_d\,\delta(\tau)\ , \tag{A-13}$$

which is incorrect. The flaw is that the use of $\delta(\tau)$, which is tantamount to $W \rightarrow \infty$, must be accompanied by having let $f_o \rightarrow \infty$, in addition. That is, $\exp(i2\pi f_o\tau)$ in (A-12) must vary as fast as $R_{\underline{n}}(\tau)$ in order for the narrowband representation to be valid. The correct end result for $f_o = W/2$ is $R_n(\tau) \rightarrow N_d\,\delta(\tau)$ as $W \rightarrow \infty$.

APPENDIX B. CHARACTERISTIC FUNCTION OF QUADRATIC AND LINEAR FORM IN CORRELATED NONZERO-MEAN GAUSSIAN RANDOM VARIABLES

The N×1 real vector X is composed of correlated Gaussian components with mean vector E and covariance matrix C; that is,

$$\overline{X} = E \,, \quad \mathrm{Cov}(X) = \overline{(X - E)(X - E)^T} = C \,. \tag{B-1}$$

Covariance matrix C is N×N, real, symmetric, and nonnegative definite. Real N×1 vector E is completely arbitrary.

We shall be interested in obtaining the characteristic function of the quadratic and linear real form in X given by

$$q = q(X) = X^T B X + 2 A^T X \,, \tag{B-2}$$

where N×N matrix B is real, symmetric, and positive definite, while real N×1 vector A is completely arbitrary. By completing the square, (B-2) can be written in the alternative form

$$q = (X + B^{-1} A)^T B (X + B^{-1} A) - A^T B^{-1} A \,. \tag{B-3}$$

Therefore, the minimum possible value of q is $- A^T B^{-1} A$. The case of an indefinite matrix B is undertaken below (B-42).

Random vector X is composed of correlated Gaussian random variables; its probability density function is [20; section 8-3]

$$p(X) = (2\pi)^{-N/2} (\det C)^{-1/2} \exp\left[- \tfrac{1}{2}(X - E)^T C^{-1} (X - E)\right] . \tag{B-4}$$

The characteristic function of interest is then given by

$$f_q(\xi) = \overline{\exp(i\xi q)} = \int dX\; p(X) \exp[i\xi q(X)] \,. \tag{B-5}$$

For convenience, we introduce variable

$$z = i2\xi \ . \tag{B-6}$$

Substitution of (B-2) and (B-4) in (B-5) yields

$$f_q(\xi) = (2\pi)^{-N/2} (\det C)^{-1/2} \times$$

$$\times \int dX \exp\left[- \tfrac{1}{2}(X - E)^T C^{-1} (X - E) + \tfrac{1}{2}z X^T B X + z A^T X\right] =$$

$$= (2\pi)^{-N/2} (\det C)^{-1/2} \times$$

$$\times \int dX \exp\left[- \tfrac{1}{2}X^T(C^{-1} - zB)X + (C^{-1} E + zA)^T X - \tfrac{1}{2}E^T C^{-1} E\right] \ . \tag{B-7}$$

Now we have the result [20; section 8-3]

$$\int dX \exp\left[- \tfrac{1}{2}X^T U X + V^T X\right] = (2\pi)^{N/2} (\det U)^{-1/2} \exp\left[\tfrac{1}{2}V^T U^{-1} V\right] \ . \tag{B-8}$$

This enables the reduction of the integral in (B-7) to

$$f_q(\xi) = (\det C)^{-1/2} \left(\det(C^{-1} - zB)\right)^{-1/2} \times$$

$$\times \exp\left[\tfrac{1}{2}(C^{-1} E + zA)^T(C^{-1} - zB)^{-1}(C^{-1} E + zA) - \tfrac{1}{2}E^T C^{-1} E\right] \ . \tag{B-9}$$

Although closed form, (B-9) is not too useful numerically because it requires an inverse of matrix $C^{-1} - zB$ for each new value of z ($= i2\xi$) of interest.

A much more compact and useful form of (B-9) can be obtained by means of the following procedure. For the given covariance matrix C and quadratic-form matrix B, solve the <u>generalized</u> characteristic-value equation

$$C\, Q = B^{-1}\, Q\, \Lambda \tag{B-10}$$

for N×N eigenvalue matrix Λ and normalized modal matrix Q, where

$$\Lambda = \mathrm{diag}[\lambda_1 \, \ldots \, \lambda_N] \; , \quad Q = [V_1 \, \ldots \, V_N] \; , \quad V_n = [v_{n1} \, \ldots \, v_{nN}]^T \; , \tag{B-11}$$

and N×1 column vector V_n is the n-th eigenvector with scalar components $\{v_{np}\}$. A procedure for obtaining the solutions Λ and Q to (B-10) is given in appendix E, when matrix B is positive definite.

(It should be noted that (B-10) does not have exactly the same solutions as the equation $B\, C\, Q' = Q'\, \Lambda'$; in fact, B C is not generally symmetric, even if B is diagonal [18; page 79, (253)]. The connection is $\Lambda' = \Lambda$, $Q' = Q\, D$, with D diagonal.)

Then, we have the two very important properties [18; pages 74 - 77] of solutions Q and Λ:

$$Q^T\, B^{-1}\, Q = I \; , \quad Q^T\, C\, Q = \Lambda \; . \tag{B-12}$$

By means of these two relations, a number of simplifications of (B-9) are possible. We begin by observing that

$$B^{-1} = Q^{-T}\, Q^{-1} \; , \; B = Q\, Q^T \; , \; C = Q^{-T}\, \Lambda\, Q^{-1} \; , \; C^{-1} = Q\, \Lambda^{-1}\, Q^T . \tag{B-13}$$

(Notice that $Q^T\, Q \neq I$.) We now employ (B-13) in (B-9) to get

$$C^{-1} - zB = Q\, (\Lambda^{-1} - zI)\, Q^T \; , \; (C^{-1} - zB)^{-1} = Q^{-T}\, (\Lambda^{-1} - zI)^{-1}\, Q^{-1} . \tag{B-14}$$

At the same time, using (B-13),

$$\det(C)\ \det(C^{-1} - z\,B) = \det(I - z\,B\,C) = \det(I - z\,Q\,\Lambda\,Q^{-1}) =$$

$$= \det(I - z\,\Lambda) = \prod_{n=1}^{N}\left(1 - z\,\lambda_n\right) . \qquad \text{(B-15)}$$

Meanwhile, if we denote twice the argument of the exponential in (B-9) by t, we have, by means of (B-14), the alternative expression

$$t = (C^{-1}\,E + zA)^T\,Q^{-T}\,(\Lambda^{-1} - zI)^{-1}\,Q^{-1}\,(C^{-1}\,E + zA) - E^T\,C^{-1}\,E . \qquad \text{(B-16)}$$

Now, by means of (B-13), develop the term

$$Q^{-1}\,(C^{-1}\,E + zA) = Q^{-1}\,C^{-1}\,E + zQ^{-1}\,A = \Lambda^{-1}\,Q^T\,E + zQ^T\,B^{-1}\,A =$$

$$= \Lambda^{-1}\,\underline{E} + z\underline{A} , \qquad \text{(B-17)}$$

where we have defined N×1 vectors

$$\underline{E} \equiv Q^T\,E = [\varepsilon_1\ \dots\ \varepsilon_N]^T , \quad \varepsilon_n = V_n^T\,E ,$$

$$\underline{A} \equiv Q^T\,B^{-1}\,A = [\alpha_1\ \dots\ \alpha_N]^T , \quad \alpha_n = V_n^T\,B^{-1}\,A . \qquad \text{(B-18)}$$

Then t in (B-16) becomes, again using (B-13),

$$t = (\Lambda^{-1}\,\underline{E} + z\underline{A})^T\,(\Lambda^{-1} - zI)^{-1}\,(\Lambda^{-1}\,\underline{E} + z\underline{A}) - E^T\,Q\,\Lambda^{-1}\,Q^T\,E =$$

$$= (\Lambda^{-1}\,\underline{E} + z\underline{A})^T\,(\Lambda^{-1} - zI)^{-1}\,(\Lambda^{-1}\,\underline{E} + z\underline{A}) - \underline{E}^T\,\Lambda^{-1}\,\underline{E} =$$

$$= \sum_{n=1}^{N} \frac{\left(\varepsilon_n/\lambda_n + z\,\alpha_n\right)^2}{1/\lambda_n - z} - \sum_{n=1}^{N} \frac{\varepsilon_n^2}{\lambda_n} =$$

$$= z \sum_{n=1}^{N} \frac{\left(\varepsilon_n + \alpha_n\right)^2}{1 - z\,\lambda_n} - z \sum_{n=1}^{N} \alpha_n^2 . \qquad \text{(B-19)}$$

By collecting all these results together, we can express the characteristic function in (B-9) in the compact form

$$f_q(\xi) = \left[\prod_{n=1}^{N}\left(1-i2\xi\lambda_n\right)\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{n=1}^{N} \frac{\left(\varepsilon_n + \alpha_n\right)^2}{1 - i2\xi\lambda_n} - i\xi \sum_{n=1}^{N} \alpha_n^2\right] . \quad \text{(B-20)}$$

The last constant in (B-20) has the alternative representation

$$- \sum_{n=1}^{N} \alpha_n^2 = - \underline{A}^T \underline{A} = - A^T B^{-1} Q Q^T B^{-1} A = - A^T B^{-1} A , \quad \text{(B-21)}$$

where we used (B-18) and (B-13); this is just the residual constant encountered in (B-3). Thus, the characteristic function of the leading (nonnegative) quadratic form in (B-3) is just (B-20) without the second summation inside the exponential. Additional relations available from (B-18) and (B-13) are

$$\sum_{n=1}^{N} \varepsilon_n^2 = \underline{E}^T \underline{E} = E^T Q Q^T E = E^T B E ,$$

$$\sum_{n=1}^{N} \varepsilon_n \alpha_n = \underline{E}^T \underline{A} = E^T Q Q^T B^{-1} A = E^T A = A^T E . \quad \text{(B-22)}$$

However, we need the constants $\{\alpha_n\}$ and $\{\varepsilon_n\}$ individually for characteristic function (B-20). The quantities $\{2\lambda_n\}$ and $\{(\varepsilon_n + \alpha_n)^2\}$ should be computed once and stored in arrays prior to the computation of $f_q(\xi)$ in (B-20) at the numerous ξ values required.

Strictly, the leading product in (B-20) is a product of N principal-value square roots. However, it can be evaluated

numerically as a single square root of a product of the N terms $\{1-i2\xi\lambda_n\}$, provided that the location of this product is tracked in the complex plane from the point 1 when $\xi = 0$. See, for example, [21; page B-5, lines 220 - 270].

If the linear form in (B-2) is absent, then vector A = 0, $\underline{A} = 0$, and $\alpha_n = 0$ for all n. However, we still need quantities

$$\varepsilon_n = V_n^T \, E \quad \text{for } 1 \le n \le N \,, \tag{B-23}$$

for the exponent in (B-20), thereby necessitating calculation of eigenvectors $\{V_n\}$. Only when mean vector E is also zero (in addition to A) do just the eigenvalues $\{\lambda_n\}$ of (B-10) suffice for calculation of characteristic function $f_q(\xi)$ in (B-20).

In this latter case, the matrix B C can be considered instead, since it has the same eigenvalues $\{\lambda_n\}$. This follows from the following manipulations: (B-10) and (B-11) can be written as

$$C \, [V_1 \; . \; . \; . \; V_N] = B^{-1} \, [V_1 \; . \; . \; . \; V_N] \, \mathrm{diag}[\lambda_1 \; . \; . \; . \; \lambda_N] \,, \tag{B-24}$$

or

$$C \, V_n = B^{-1} \, V_n \, \lambda_n \quad \text{for } 1 \le n \le N \,. \tag{B-25}$$

Therefore

$$0 = (C - B^{-1} \, \lambda_n) \, V_n = B^{-1} \, (B \, C - \lambda_n \, I) \, V_n \tag{B-26}$$

or, since B is positive definite,

$$\det(B \, C - \lambda_n \, I) = 0 \quad \text{for } 1 \le n \le N \,. \tag{B-27}$$

Thus, $\{\lambda_n\}$ are the eigenvalues of matrix B C as well as (B-10).

CUMULANTS OF q

Before determining the cumulants of q, we present a few relations which will enable easier manipulations of various matrices encountered below. From (B-13), there follows

$$B\ C = Q\ \Lambda\ Q^{-1}\ , \quad B\ C\ B = Q\ \Lambda\ Q^{T}\ ,$$

$$B\ C\ B\ C = Q\ \Lambda^{2}\ Q^{-1}\ , \quad B\ C\ B\ C\ B = Q\ \Lambda^{2}\ Q^{T}\ . \qquad \text{(B-28)}$$

Therefore

$$\mathrm{tr}(B\ C) = \mathrm{tr}(\Lambda\ Q^{-1}\ Q) = \mathrm{tr}(\Lambda) = \sum_{n=1}^{N} \lambda_n\ ,$$

$$\mathrm{tr}\left((B\ C)^{2}\right) = \mathrm{tr}(\Lambda^{2}\ Q^{-1}\ Q) = \mathrm{tr}(\Lambda^{2}) = \sum_{n=1}^{N} \lambda_n^{2}\ . \qquad \text{(B-29)}$$

Now, if we expand the natural logarithm of the general characteristic function $f_q(\xi)$ in (B-20) in a power series in $i\xi$, we can easily pick off the cumulants as

$$\chi_q(1) = \mu_q = \sum_{n=1}^{N}\left(\lambda_n + \varepsilon_n^{2} + 2\varepsilon_n\alpha_n\right) = \mathrm{tr}(B\ C) + E^{T}\ B\ E + 2\ A^{T}\ E\ ,$$

$$\chi_q(p) = 2^{p-1}\ (p-1)!\ \sum_{n=1}^{N} \lambda_n^{p-1}\left(\lambda_n + p(\varepsilon_n + \alpha_n)^{2}\right) \text{ for } p \geq 2. \qquad \text{(B-30)}$$

Here, we used (B-29) and (B-22). In particular, the variance of q is

$$\chi_q(2) = \sigma_q^{2} = 2\sum_{n=1}^{N}\left(\lambda_n^{2} + 2\lambda_n(\varepsilon_n + \alpha_n)^{2}\right) =$$

$$= 2\ \mathrm{tr}\left((B\ C)^{2}\right) + 4\ (B\ E + A)^{T}\ C\ (B\ E + A)\ . \qquad \text{(B-31)}$$

This latter relation follows from these manipulations:

$$\sum_{n=1}^{N} \lambda_n (\varepsilon_n + \alpha_n)^2 = (\underline{E} + \underline{A})^T \Lambda (\underline{E} + \underline{A}) =$$

$$= (E + B^{-1}A)^T Q \Lambda Q^T (E + B^{-1}A) = (E + B^{-1}A)^T B C B (E + B^{-1}A) =$$

$$= (B E + A)^T C (B E + A) . \tag{B-32}$$

Here, we used (B-18) and (B-28).

ALTERNATIVE DERIVATION OF CHARACTERISTIC FUNCTION (B-20)

We start again with conditions (B-1) and (B-2). Then, solve (B-10) for Q and Λ, as before. Now, consider the linear transformation of Gaussian random vector X according to

$$Y = Q^T (X - E) = [y_1 \; . \; . \; . \; y_N]^T . \tag{B-33}$$

Then N×1 Gaussian vector Y has mean $\overline{Y} = 0$ and covariance matrix

$$\overline{Y \, Y^T} = Q^T \, \overline{(X - E)(X - E)^T} \, Q = Q^T C Q = \Lambda , \tag{B-34}$$

where we used (B-1) and (B-12). This diagonal matrix means that

$$\overline{y_n} = 0 , \quad \overline{y_n \, y_m} = \lambda_n \, \delta_{nm} ; \tag{B-35}$$

that is, $\{y_n\}$ are zero-mean uncorrelated (and therefore <u>independent</u>) Gaussian random variables. This is the key to this development.

At the same time, solving (B-33) for X, we have, with (B-12),

$$X = E + Q^{-T}\,Y = E + B^{-1}\,Q\,Y\,. \tag{B-36}$$

Then quadratic form (B-2) can be expressed as

$$\begin{aligned} q &= (E + B^{-1}\,Q\,Y)^T\,B\,(E + B^{-1}\,Q\,Y) + 2\,A^T\,(E + B^{-1}\,Q\,Y) = \\ &= Y^T\,Y + 2\,(E^T\,Q + A^T\,B^{-1}\,Q)\,Y + E^T\,B\,E + 2\,A^T\,E = \\ &= Y^T\,Y + 2\,(\underline{E}^T + \underline{A}^T)\,Y + E^T\,B\,E + 2\,A^T\,E = \\ &= \sum_{n=1}^{N} \left(y_n^2 + 2(\varepsilon_n + \alpha_n)y_n + \varepsilon_n^2 + 2\varepsilon_n\alpha_n\right)\,, \end{aligned} \tag{B-37}$$

where we used (B-12), (B-18), and (B-22). Therefore, using the independence property derived in (B-34) and (B-35), the characteristic function of q is

$$\begin{aligned} f_q(\xi) = \overline{\exp(i\xi q)} &= \prod_{n=1}^{N} \left\{ \int \frac{dy_n}{\left(2\pi\lambda_n\right)^{\frac{1}{2}}} \exp\left(-\frac{y_n^2}{2\lambda_n}\right) \times \right. \\ &\left. \times \exp\left[i\xi\left(y_n^2 + 2(\varepsilon_n + \alpha_n)y_n + \varepsilon_n^2 + 2\varepsilon_n\alpha_n\right)\right]\right\} = \\ &= \prod_{n=1}^{N} \left\{ \left(1 - i2\xi\lambda_n\right)^{-\frac{1}{2}} \exp\left[i\xi\,\frac{(\varepsilon_n + \alpha_n)^2}{1 - i2\xi\lambda_n} - i\xi\alpha_n^2\right]\right\}\,, \end{aligned} \tag{B-38}$$

which is equal to (B-20).

PROPERTIES OF EIGENVECTORS $\{V_n\}$

Consider the representation of modal matrix Q in terms of eigenvectors $\{V_n\}$ in (B-11). Then the first relation in (B-12) and the second relation in (B-13) yield, respectively,

$$V_n^T B^{-1} V_m = \delta_{nm} , \quad \sum_{n=1}^{N} V_n V_n^T = B . \tag{B-39}$$

Similarly, the second relation in (B-12) and the fourth relation in (B-13) yield, respectively,

$$V_n^T C V_m = \lambda_n \delta_{nm} , \quad \sum_{n=1}^{N} \frac{1}{\lambda_n} V_n V_n^T = C^{-1} . \tag{B-40}$$

Thus, there are two orthogonality relations satisfied by the eigenvectors $\{V_n\}$, namely the leading relations in (B-39) and (B-40). If, in addition, matrix B or C is diagonal, then the trailing relations in (B-39) and (B-40) yield an additional orthogonality property.

In terms of the components (B-11), the relations above become

$$\sum_{k,j=1}^{N} v_{nk} \left(B^{-1}\right)_{kj} v_{mj} = \delta_{nm} , \quad \sum_{n=1}^{N} v_{nk} v_{nj} = \left(B\right)_{kj} , \tag{B-41}$$

and

$$\sum_{k,j=1}^{N} v_{nk} \left(C\right)_{kj} v_{mj} = \lambda_n \delta_{nm} , \quad \sum_{n=1}^{N} \frac{1}{\lambda_n} v_{nk} v_{nj} = \left(C^{-1}\right)_{kj} . \tag{B-42}$$

GENERAL SYMMETRIC MATRIX B

All the results above are based on the premise that matrix B in the quadratic form in (B-2) is positive definite. However, there are examples where this is not the situation, in which case the corresponding procedure for solution of (B-10) in appendix E (that was mentioned under (B-11)) is not applicable. We now give an alternative procedure for obtaining the characteristic function of q in (B-2) for <u>any</u> real matrix B, whether definite or not. Matrix B can be taken symmetric without loss of generality, since only the symmetric part of B is active in (B-2).

We first observe that the N×N covariance matrix C in (B-1) is always nonnegative definite because

$$V^T\, C\, V = V^T\, \overline{(X - E)(X - E)^T}\, V = \overline{\left(V^T(X - E)\right)^2} \geq 0 \qquad \text{(B-43)}$$

for any N×1 real vector V. We shall presume that C is positive definite. Instead of solving (B-10), we solve the alternative generalized real characteristic-value matrix equation

$$B\, Q = C^{-1}\, Q\, \Lambda \qquad \text{(B-44)}$$

for N×N normalized modal matrix Q and diagonal eigenvalue matrix Λ. A procedure for this solution is given in appendix E; see (E-14) and sequel.

Then, these solutions satisfy [18; pages 74 - 77]

$$Q^T\, C^{-1}\, Q = I\ , \qquad \text{(B-45)}$$

$$Q^T\, B\, Q = \Lambda \equiv \mathrm{diag}[\lambda_1\ .\ .\ .\ \lambda_N]\ . \qquad \text{(B-46)}$$

Now let linearly transformed N×1 random vector

$$Z = Q^{-1} (X - E) \equiv [z_1 \ . \ . \ . \ z_N]^T \ . \tag{B-47}$$

Then mean $\overline{Z} = 0$ and covariance matrix

$$\overline{Z \ Z^T} = Q^{-1} \ \overline{(X - E)(X - E)^T} \ Q^{-T} = Q^{-1} \ C \ Q^{-T} = I \ , \tag{B-48}$$

upon use of (B-1) and (B-45). Solve (B-47) for X, obtaining

$$X = Q \ Z + E \ . \tag{B-49}$$

Now, substitute (B-49) into quadratic and linear form (B-2):

$$q = (Q \ Z + E)^T \ B \ (Q \ Z + E) + 2 \ A^T \ (Q \ Z + E) =$$

$$= (Z + Q^{-1} \ E)^T \ Q^T \ B \ Q \ (Z + Q^{-1} \ E) + 2 \ A^T \ Q \ (Z + Q^{-1} \ E) =$$

$$= (Z + \underline{E})^T \ \Lambda \ (Z + \underline{E}) + 2 \ \underline{A}^T \ (Z + \underline{E}) \ , \tag{B-50}$$

where we used (B-46) and defined two auxiliary N×1 vectors

$$\underline{E} = Q^{-1} \ E \equiv [\varepsilon_1 \ . \ . \ . \ \varepsilon_N]^T \tag{B-51}$$

and

$$\underline{A} = Q^T \ A \equiv [\alpha_1 \ . \ . \ . \ \alpha_N]^T \ . \tag{B-52}$$

Therefore, (B-50) can now be expressed as

$$q = \sum_{n=1}^{N} (\lambda_n \ z_n + \lambda_n \ \varepsilon_n + 2 \ \alpha_n)(z_n + \varepsilon_n) \ . \tag{B-53}$$

The characteristic function of q can now be readily found, with the assistance of covariance property (B-48), in its most compact form

$$f_q(\xi) = \overline{\exp(i\xi q)} =$$

$$= \prod_{n=1}^{N} \left(\int dz_n \; (2\pi)^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2} z_n^2 + i\xi \left(\lambda_n z_n + \lambda_n \varepsilon_n + 2\alpha_n \right) \left(z_n + \varepsilon_n \right) \right] \right) =$$

$$= \left[\prod_{n=1}^{N} \left(1 - i2\xi\lambda_n \right) \right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{n=1}^{N} \frac{\lambda_n \varepsilon_n^2 + 2\varepsilon_n \alpha_n + i2\xi\alpha_n^2}{1 - i2\xi\lambda_n} \right] . \qquad \text{(B-54)}$$

In summary, the following computations must be performed: solve (B-44) for N×N matrices Q and Λ; compute N×1 vectors $\underline{E}$ and $\underline{A}$ by means of (B-51) and (B-52), respectively; and evaluate $f_q(\xi)$ at desired ξ values by use of (B-54). The only inverse matrix required is that of modal matrix Q; the solution of (B-44) does not actually require calculation of C^{-1}, as will be seen in appendix E, (E-14) and sequel.

If mean vector E in (B-1) is zero and/or if linear form vector A in (B-2) is zero, the corresponding calculations in (B-51) and (B-52) can be circumvented. If both E and A are zero, the exp term in (B-54) is absent altogether.

In this latter case, only the eigenvalues $\{\lambda_n\}$ of (B-44) need be determined. If we let V_n be the n-th eigenvector (column) of Q, equation (B-44) takes the form

$$B\,V_n = C^{-1}\,V_n\,\lambda_n \quad \text{for } 1 \le n \le N . \qquad \text{(B-55)}$$

This can be manipulated into

$$(B - \lambda_n\,C^{-1})V_n = 0 , \qquad \text{(B-56)}$$

meaning that

$$\det(B - \lambda_n C^{-1}) = 0 \qquad \text{(B-57)}$$

or, since $\det(C) \neq 0$, that

$$\det(B\,C - \lambda_n I) = 0 \;. \qquad \text{(B-58)}$$

That is, $\{\lambda_n\}$ are the eigenvalues of matrix B C. This is a much simpler numerical task than solving (B-44) for both Q and Λ.

If none of the eigenvalues $\{\lambda_n\}$ are zero or near zero, then (B-46) furnishes inverse matrix $Q^{-1} = \Lambda^{-1}\, Q^T\, B$ and an alternative to (B-51), namely

$$\underline{E} = \Lambda^{-1}\, Q^T\, B\, E \;,$$

$$\varepsilon_n = \frac{1}{\lambda_n}\, V_n^T\, B\, E \quad \text{for } 1 \leq n \leq N \;. \qquad \text{(B-59)}$$

Also, from (B-52), regardless of the sizes of the eigenvalues,

$$\alpha_n = V_n^T\, A \quad \text{for } 1 \leq n \leq N \;. \qquad \text{(B-60)}$$

The interrelationships between Q and Λ in (B-44) and the corresponding matrices Q and Λ in (B-10) are derived in appendix E, (E-24) and sequel. The identity of characteristic functions (B-20) and (B-54) is also verified there.

PROPERTIES OF EIGENVECTORS $\{V_n\}$

Relations (B-45) and (B-46) yield the following orthogonality properties between the eigenvectors:

$$V_n^T\, C^{-1}\, V_m = \delta_{nm} , \qquad V_n^T\, B\, V_m = \lambda_n\, \delta_{nm} . \qquad \text{(B-61)}$$

Also, since (B-45) and (B-46) can be alternatively expressed as $C = Q\, Q^T$ and $B^{-1} = Q\, \Lambda^{-1}\, Q^T$, we have

$$\sum_{n=1}^{N} V_n\, V_n^T = C , \qquad \sum_{n=1}^{N} \frac{1}{\lambda_n}\, V_n\, V_n^T = B^{-1} . \qquad \text{(B-62)}$$

As a special case of (B-61), it can be seen that $\lambda_n = V_n^T\, B\, V_n$; therefore, if matrix B is nonnegative definite, then $\lambda_n \geq 0$ for all n. The following example demonstrates that indefinite symmetric B matrices can lead to negative eigenvalues, even when matrix C is symmetric and positive definite:

$$B = \begin{bmatrix} 4 & -6 \\ -6 & 7 \end{bmatrix} , \qquad C = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} ; \qquad \text{(B-63)}$$

the eigenvalue and eigenvector solutions to (B-44) are

$$\Lambda = \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} , \qquad Q = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} . \qquad \text{(B-64)}$$

CUMULANTS OF q

By taking the logarithm of characteristic function $f_q(\xi)$ in (B-54) and expanding in a power series in $i\xi$, the cumulants of quadratic and linear form q are found to be

$$\chi_q(m) = \left\{ \begin{array}{ll} \displaystyle\sum_{n=1}^{N} \left[\lambda_n\left(1 + \varepsilon_n^2\right) + 2\varepsilon_n\alpha_n\right] & \text{for } m = 1 \\ (m-1)!\ 2^{m-1} \displaystyle\sum_{n=1}^{N} \lambda_n^{m-2}\left[\lambda_n^2 + m\left(\lambda_n\varepsilon_n + \alpha_n\right)^2\right] & \text{for } m \geq 2 \end{array} \right\}. \quad \text{(B-65)}$$

APPENDIX C. CHARACTERISTIC FUNCTION OF OUTPUT γ FOR GENERAL TRANSMITTED SIGNAL ENERGIES $\{\tilde{E}_k\}$ AND RECEIVER WEIGHTS $\{A_k\}$

The conditional characteristic function of processor output γ, for arbitrary transmitted signal energies $\{\tilde{E}_k\}$ and receiver weights $\{A_k\}$, was given in (27) as

$$f_c(\xi) = \prod_{k=1}^{K} \left(1 - i\xi\, 2\sigma_k^2\right)^{-1} \exp\left[i\xi \sum_{k=1}^{K} w_k\, r_k^2\right] , \tag{C-1}$$

where

$$\sigma_k^2 = 2\, N_o\, A_k^2\, \tilde{E}_k , \quad w_k = \frac{4\, A_k^2\, \tilde{E}_k^2}{1 - i\xi\, 4\, N_o\, A_k^2\, \tilde{E}_k} . \tag{C-2}$$

It is important to notice that the coefficients $\{w_k\}$ in the summation in (C-1) are complex and are functions of argument ξ.

Now, we want to find the <u>unconditional</u> characteristic function of γ, after averaging over the statistics of amplitude scalings $\{r_k\}$. From (43) - (45),

$$r_k^2 = \sum_{m=1}^{M} \left[c_m(t_k,f_k) + g_m(t_k,f_k)\right]^2 = \sum_{m=1}^{M} x_{mk}^2 , \tag{C-3}$$

where Gaussian random variables

$$x_{mk} = c_m(t_k,f_k) + g_m(t_k,f_k) \quad \text{for } 1 \le m \le M , \quad 1 \le k \le K . \tag{C-4}$$

Therefore, the exponential term in (C-1) becomes

$$\exp\left[i\xi \sum_{k=1}^{K} w_k\, r_k^2\right] = \exp\left[i\xi\, X^T\, \tilde{W}\, X\right] , \tag{C-5}$$

where MK×1 random Gaussian real column vector

$$X = [x_{11} \cdots x_{M1} \; \cdots \; x_{1k} \cdots x_{Mk} \; \cdots \; x_{1K} \cdots x_{MK}]^T \; , \qquad \text{(C-6)}$$

and MK×MK diagonal matrix of complex elements

$$\tilde{W} = \text{diag}[w_1 \cdots w_1 \; \cdots \; \underbrace{w_k \cdots w_k}_{\text{M terms}} \; \cdots \; w_K \cdots w_K \;] \; . \qquad \text{(C-7)}$$

The MK×MK covariance matrix of vector X is

$$K_x = \text{Cov}\{X\} = \overline{(X - \overline{X}) \; (X - \overline{X})^T} \qquad \text{(C-8)}$$

which is known.

The unconditional characteristic function of γ is available from (C-1) and (C-5) in the form

$$f_\gamma(\xi) = \overline{f_c(\xi)} = \prod_{k=1}^{K} \left(1 - i\xi \; 2\sigma_k^2\right)^{-1} \overline{\exp\left[i\xi \; X^T \; \tilde{W} \; X\right]} =$$

$$= \prod_{k=1}^{K} \left(1 - i\xi \; 2\sigma_k^2\right)^{-1} \int dX \; (2\pi)^{-MK/2} \left(\det K_x\right)^{-\frac{1}{2}} \times$$

$$\times \exp\left[- \tfrac{1}{2}(X - \overline{X})^T \; K_x^{-1} \; (X - \overline{X}) + i\xi \; X^T \; \tilde{W} \; X\right] =$$

$$= \prod_{k=1}^{K} \left(1 - i\xi \; 2\sigma_k^2\right)^{-1} \left(\det\left(I - i\xi 2 \; \tilde{W} \; K_x\right)\right)^{-\frac{1}{2}} \times$$

$$\times \exp\left[\tfrac{1}{2} \; \overline{X}^T \; (I - i\xi 2 \; \tilde{W} \; K_x)^{-1} \; K_x^{-1} \; \overline{X} - \tfrac{1}{2} \; \overline{X}^T \; K_x^{-1} \; \overline{X}\right] \; . \qquad \text{(C-9)}$$

The matrix $\tilde{W}$ depends on ξ through coefficients $\{w_k\}$ in (C-2). Therefore, in order to compute (C-9), it is necessary to invert a complex MK×MK matrix for each ξ of interest. However, if mean $\overline{X}$ is zero, only the determinant in (C-9) need be evaluated.

UNCORRELATED FADING COMPONENTS $\{g_m(t,f)\}$

In this subsection, we presume that the fading components in model (45) are uncorrelated and therefore independent. This means that, in (C-4), random variables $x_{m1},\ldots,x_{mK}$ are independent of $x_{n1},\ldots,x_{nK}$ for $m \neq n$. Reference to (C-1) and (C-3) then yields the characteristic function of output γ in the form

$$f_\gamma(\xi) = \overline{f_c(\xi)} = \prod_{k=1}^{K}\left(1 - i\xi\, 2\sigma_k^2\right)^{-1} E(\xi) \ , \tag{C-10}$$

where

$$E(\xi) \equiv \overline{\exp\left[i\xi \sum_{k=1}^{K} w_k\, r_k^2\right]} = \overline{\exp\left[i\xi \sum_{k=1}^{K} w_k \sum_{m=1}^{M} x_{mk}^2\right]} =$$

$$= \prod_{m=1}^{M} \overline{\exp\left[i\xi \sum_{k=1}^{K} w_k\, x_{mk}^2\right]} \ . \tag{C-11}$$

Now, let K×1 real random vector X_m and K×K complex diagonal matrix W be defined according to

$$X_m = [x_{m1} \cdot \cdot \cdot x_{mK}]^T \ , \quad W = \text{diag}[w_1 \cdot \cdot \cdot w_K] \ . \tag{C-12}$$

Also, let the covariance matrix of X_m be denoted by K_m, which is a K×K real matrix, for $1 \leq m \leq M$. Then, the m-th term in (C-11) becomes

$$\overline{\exp\left[i\xi \sum_{k=1}^{K} w_k\, x_{mk}^2\right]} = \overline{\exp\left[i\xi\, X_m^T\, W\, X_m\right]} = \int dX_m\, (2\pi)^{-K/2} \times$$

$$\times\ (\det K_m)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(X_m - \overline{X}_m\right)^T K_m^{-1}\left(X_m - \overline{X}_m\right) + i\xi\, X_m^T\, W\, X_m\right] =$$

$$\left(\det\left(I-i\xi 2WK_m\right)\right)^{-\frac{1}{2}} \exp\left[\frac{1}{2}\, \overline{X}_m^T \left(I - i\xi 2\, W\, K_m\right)^{-1} K_m^{-1}\, \overline{X}_m - \frac{1}{2}\, \overline{X}_m^T\, K_m^{-1}\, \overline{X}_m\right] \tag{C-13}$$

Finally, the desired unconditional characteristic function of processor output γ follows from a combination of (C-10), (C-11), and (C-13) as

$$f_\gamma(\xi) = \prod_{k=1}^{K}\left(1 - i\xi\; 2\sigma_k^2\right)^{-1} \prod_{m=1}^{M}\Bigg\{\left(\det\left(I - i\xi 2\; W\; K_m\right)\right)^{-\frac{1}{2}} \times$$
$$\times \exp\left[\frac{1}{2}\, \overline{X}_m^T \left(I - i\xi 2\; W\; K_m\right)^{-1} K_m^{-1}\, \overline{X}_m - \frac{1}{2}\, \overline{X}_m^T\, K_m^{-1}\, \overline{X}_m\right]\Bigg\} \;. \tag{C-14}$$

The major computational burden here consists of the inverses of M complex K×K matrices for each ξ value of interest. However, if the M covariance matrices $\{K_m\}$ are all identical, the task is considerably simplified. Alternatively, if all the means $\{\overline{X}_m\}$ are zero, then the evaluation is limited to M determinants of the complex M×M matrices $I - i\xi 2\; W\; K_m$. The K×K complex matrix W is defined by (C-12) and (C-2).

APPENDIX D. DERIVATION OF PROBABILITY DENSITY AND EXCEEDANCE DISTRIBUTION FROM CHARACTERISTIC FUNCTION

The general form in (136) and (158) of the characteristic function of processor output γ can be written as

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi\left(1 + T_{mk}\right)\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{e_{mk}}{1 - i2\xi\left(1 + T_{mk}\right)}\right] , \tag{D-1}$$

where

$$T_{mk} = \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)} \quad \text{and} \quad e_{mk} = \underline{\varepsilon}_k^{(m)^2} . \tag{D-2}$$

We want to determine the corresponding probability density function and exceedance distribution function. There is a possible numerical problem using Fourier transforms on (D-1) directly, since its asymptotic decay is only proportional to ξ^{-K} as $\xi \to \infty$, which makes accurate results difficult to achieve for small K.

We will expand (D-1) in a power series in $(1 - i2\xi)^{-1}$ and thereby obtain not only an asymptotic expansion of $f_\gamma(\xi)$, but in fact, a convergent series which can be transformed term-by-term to yield the probability density function of γ. Integration then yields the exceedance distribution function. Furthermore, it will turn out that all the series coefficients are positive and related by recursions involving only positive terms. These properties enable very accurate and efficient evaluation.

We begin by making the transformation

$$z = \frac{1}{1 - i2\xi}\ , \quad i2\xi = -\ \frac{1 - z}{z}\ . \tag{D-3}$$

Substitution in (D-1) yields, after manipulation,

$$f_\gamma(\xi) = F\ z^K\ D(z)\ E(z)\ , \tag{D-4}$$

where (using obvious abbreviations)

$$r_{mk} = \frac{T_{mk}}{1 + T_{mk}}\ , \quad F = \left[\prod_{m,k}^{M,K}\left(1 + T_{mk}\right)\right]^{-\frac{1}{2}}\ , \tag{D-5}$$

$$D(z) = \left[\prod_{m,k}^{M,K}\left(1 - z\ r_{mk}\right)\right]^{-\frac{1}{2}},\quad E(z) = \exp\left[-\ \frac{1 - z}{2}\sum_{m,k}^{M,K}\frac{e_{mk}/(1+T_{mk})}{1 - z\ r_{mk}}\right]\ .$$

To develop the series of D(z) in powers of z, expand

$$\ln D(z) = -\ \frac{1}{2}\sum_{m,k}^{M,K}\ln(1 - z\ r_{mk}) = \sum_{p=1}^{\infty}\beta_p\ z^p\ , \tag{D-6}$$

where

$$\beta_p = \frac{1}{2p}\sum_{m,k}^{M,K} r_{mk}^p \quad \text{for } p \ge 1\ . \tag{D-7}$$

Notice that $\beta_p \ge 0$ for $p \ge 1$ since $T_{mk} \ge 0$; this latter property follows from (D-2) and the fact that eigenvalues of covariance matrices can never be negative.

At the same time, the expansion of ln E(z) in (D-5) is

$$\ln E(z) = -\ \frac{1}{2}(1 - z)\sum_{m,k}^{M,K}\frac{e_{mk}}{1 + T_{mk}}\sum_{p=0}^{\infty} r_{mk}^p\ z^p\ . \tag{D-8}$$

By combining the results in (D-6) - (D-8), there follows, after some manipulations,

$$\ln(D(z)\ E(z)) = \sum_{p=0}^{\infty} \alpha_p\ z^p \ , \tag{D-9}$$

where

$$\alpha_o = -\frac{1}{2} \sum_{m,k}^{M,K} \frac{e_{mk}}{1 + T_{mk}} \ ,$$

$$\alpha_p = \frac{1}{2} \sum_{m,k}^{M,K} \frac{T_{mk}^{p-1}}{(1 + T_{mk})^p} \left[\frac{T_{mk}}{p} + \frac{e_{mk}}{1 + T_{mk}}\right] \quad \text{for } p \geq 1 \ . \tag{D-10}$$

Notice that $\alpha_p \geq 0$ for $p \geq 1$.

The desired product for use in (D-4) is then [17; page 93]

$$D(z)\ E(z) = \exp\left(\sum_{p=0}^{\infty} \alpha_p\ z^p\right) = \sum_{p=0}^{\infty} g_p\ z^p \ , \tag{D-11}$$

where

$$g_o = \exp(\alpha_o) \ , \qquad g_p = \frac{1}{p} \sum_{n=1}^{p} n\ \alpha_n\ g_{p-n} \quad \text{for } p \geq 1 \ . \tag{D-12}$$

It is important to notice that $g_p \geq 0$ for $p \geq 0$; that is, the recursion in (D-12) involves no negative quantities, thereby avoiding cancellation error.

Finally, the characteristic function in (D-4) can be expressed in the desired power series in z $(= (1 - i2\xi)^{-1})$ as

$$f_\gamma(\xi) = F\ z^K \sum_{p=0}^{\infty} g_p\ z^p \ . \tag{D-13}$$

Now, we know the following Fourier transform pair between a characteristic function and a probability density function:

$$z^n = \frac{1}{(1 - i2\xi)^n} \longleftrightarrow \frac{u^{n-1} \exp(-u/2)}{2^n \ (n-1)!} \quad \text{for } u > 0 \ . \qquad \text{(D-14)}$$

This enables us to write the probability density function of γ, directly from (D-13), in the form

$$p_\gamma(u) = F \sum_{p=0}^{\infty} g_p \frac{u^{K+p-1} \exp(-u/2)}{2^{K+p} \ (K+p-1)!} \quad \text{for } u > 0 \ , \qquad \text{(D-15)}$$

where scale factor F is given by (D-5). All the terms in this series are nonnegative.

Integration on the positive tail of density p_γ gives the exceedance distribution function of γ:

$$\text{Prob}(\gamma > u) = \int_u^{\infty} dt \ p_\gamma(t) = F \sum_{p=0}^{\infty} g_p \ H_{K+p-1}\left(\frac{u}{2}\right) \quad \text{for } u > 0 \ , \ \text{(D-16)}$$

where F is given by (D-5) and

$$H_n(x) \equiv \exp(-x) \sum_{k=0}^{n} \frac{x^k}{k!} \quad \text{for } n \geq 0 \ , \ x \geq 0 \ . \qquad \text{(D-17)}$$

This latter sequence of functions is easily generated by use of the coupled recurrences

$$H_o(x) = \exp(-x) \ , \quad H_n(x) = T_n(x) + H_{n-1}(x) \quad \text{for } n \geq 1 \ ,$$

$$T_o(x) = \exp(-x) \ , \quad T_n(x) = T_{n-1}(x) \ \frac{x}{n} \quad \text{for } n \geq 1 \ . \qquad \text{(D-18)}$$

The two recurrences derived above, namely (D-12) and (D-18), utilize only nonnegative quantities. The single negative term, α_o in (D-10), is immediately converted to positive quantity g_o in (D-12) and α_o is not encountered again.

Since processor output γ can never be negative (see figure 1), we have, from (D-16) and (D-17),

$$1 = \mathrm{Prob}(\gamma > 0) = F \sum_{p=0}^{\infty} g_p \ . \tag{D-19}$$

This relation can be used to furnish an upper bound on the error incurred by using up to the p = N term in series (D-16) for the exc edance distribution function. In particular, the error is

$$F \sum_{p=N+1}^{\infty} g_p \; H_{K+p-1}\left(\frac{u}{2}\right) \le F \sum_{p=N+1}^{\infty} g_p = 1 - F \sum_{p=0}^{N} g_p \tag{D-20}$$

for all $u \ge 0$. The upper bound of 1 on $H_n(x)$ follows immediately from definition (D-17). Since scale factor F and coefficients $\{g_p\}$ for $0 \le p \le N$ must be computed anyway, in order to utilize (D-16), error bound (D-20) is simple to compute. It says that the error at any u is never larger than the error at the origin.

However, (D-16) converges too slowly in some cases of large signal-to-noise ratio; only sequence $\{g_p\}$ converges to zero. Result (D-19) allows a modification to sum (D-16) which will converge much more rapidly. From (D-17), since $H_n(x) \to 1$ as $n \to \infty$, we express (D-16) as

$$\text{Prob}(\gamma > u) = F \sum_{p=0}^{\infty} g_p \left[H_{K+p-1}\left(\frac{u}{2}\right) - 1 + 1 \right] =$$

$$= 1 - F \sum_{p=0}^{\infty} g_p \left[1 - H_{K+p-1}\left(\frac{u}{2}\right) \right] . \qquad \text{(D-21)}$$

Now, both factors in the sum decay to zero as p increases. In fact,

$$1 - H_{n-1}(x) \le \exp(-x) \frac{x^n}{n!} \frac{n+1}{n+1-x} \quad \text{for } n > x . \qquad \text{(D-22)}$$

This alternative in (D-21) is utilized in the program listed at the end of this appendix. Recursions for $1 - H_n(x)$, which are obvious modifications of (D-18), are also used.

As noted at the beginning of this appendix, form (D-1) encompasses (136) and (158) in the main text. It therefore also covers (139) and (160), because these are reductions that can be obtained, respectively, by setting all $\{e_{mk}\}$ to zero. If all the $\{e_{mk}\}$ and $\{T_{mk}\}$ are zero, then all $\{\alpha_p\}$ are zero, from (D-10). Then, all $\{g_p\}$ are zero except $g_o = 1$, from (D-12). In this case of noise-only, series (D-16) has just the one term $H_{K-1}(u/2)$.

The results above, for the expansions of the characteristic function, probability density function, and exceedance distribution function, also apply directly to forms (124) (and (126)) as well as (144) (and (149)) in the main text, but with the following changes. For (124), define instead

$$T_n = \frac{2\tilde{E}}{N_o} \lambda_n , \quad e_n = \frac{2\tilde{E}}{N_o} \varepsilon_n^2 , \quad F = \left[\prod_{n=1}^{KM} \left(1 + T_n\right) \right]^{-\frac{1}{2}} , \qquad \text{(D-23)}$$

along with

$$\alpha_o = -\frac{1}{2}\sum_{n=1}^{KM} \frac{e_n}{1 + T_n} ,$$

$$\alpha_p = \frac{1}{2}\sum_{n=1}^{KM} \frac{T_n^{p-1}}{(1 + T_n)^p}\left[\frac{T_n}{p} + \frac{e_n}{1 + T_n}\right] \quad \text{for } p \geq 1 . \tag{D-24}$$

The recurrence (D-12) involving $\{g_p\}$ is unchanged.

For (144), define instead

$$T_k = \frac{2E_{11}}{N_o}\,\underline{\lambda}_k , \qquad F = \left[\prod_{k=1}^{K}\left(1 + T_k\right)\right]^{-M/2} , \tag{D-25}$$

along with

$$\alpha_o = -\frac{1}{2}\sum_{k=1}^{K} \frac{\underline{h}_k}{1 + T_k} ,$$

$$\alpha_p = \frac{1}{2}\sum_{k=1}^{K} \frac{T_k^{p-1}}{(1 + T_k)^p}\left[\frac{M}{p}\,T_k + \frac{\underline{h}_k}{1 + T_k}\right] \quad \text{for } p \geq 1 . \tag{D-26}$$

Again, the recurrence involving $\{g_p\}$ is unchanged.

REFINEMENT

In some cases, a refinement of the above procedure may be worthwhile, especially for larger signal-to-noise ratios. We illustrate it with reference to (150), which applies to $M = 2$, two components in fading model (45). In this case, we have characteristic function

$$f_Y(\xi) = \left[\prod_{k=1}^{K}\left(1 - i2\xi Q_k\right)\right]^{-1}, \tag{D-27}$$

where

$$Q_k = 1 + \frac{2E_{11}}{N_o}\,\underline{\lambda}_k \quad \text{for } 1 \le k \le K\,. \tag{D-28}$$

These quantities $\{Q_k\}$ are always larger than 1 for nonzero signal-to-noise ratio.

Instead of making transformation (D-3), we modify it to the more general form

$$z = \frac{1}{1 - i2\xi Q}, \tag{D-29}$$

where Q can be chosen larger than 1 if desired. Then, defining

$$F = Q^K \left[\prod_{k=1}^{K} Q_k\right]^{-1}, \quad t_k = \frac{Q_k - Q}{Q_k} \quad \text{for } 1 \le k \le K\,, \tag{D-30}$$

characteristic function (D-27) can be developed as

$$f_Y(\xi) = F\, z^K \left[\prod_{k=1}^{K}\left(1 - t_k z\right)\right]^{-1} = F\, z^K \exp\left(\sum_{p=1}^{\infty} \alpha_p\, z^p\right), \tag{D-31}$$

now with

$$p\ \alpha_p = \sum_{k=1}^{K} t_k^p = \sum_{k=1}^{K} \left(\frac{Q_k - Q}{Q_k}\right)^p \quad \text{for } p \geq 1 \ . \tag{D-32}$$

If we take the constant Q to have value

$$\tilde{Q} = \min_{1 \leq k \leq K} Q_k \ , \tag{D-33}$$

then all the $\{t_k\}$ and $\{\alpha_p\}$ are nonnegative, and we still have all the desirable properties listed above. The value $\tilde{Q}$ is always larger than 1; it enables (D-31) to furnish a better fit to (D-27) with fewer terms than arbitrarily forcing Q = 1. Larger values than $\tilde{Q}$ can yield still more rapidly convergent series, but at the expense of some negative $\{g_p\}$. There follows, from (D-31)

$$f_\gamma(\xi) = F\ z^K \sum_{p=0}^{\infty} g_p\ z^p = F \sum_{p=0}^{\infty} g_p\ (1 - i2\xi Q)^{-K-p} \tag{D-34}$$

for any Q; see (D-11) and (D-12) for the $\{g_p\}$. Expansion (D-34) should be compared with the moment procedure in [17; (102) and (105)], where a different basis set, namely the generalized Laguerre functions, is used for the series expansion.

The corresponding exceedance distribution function is

$$\text{Prob}(\gamma > u) = F \sum_{p=0}^{\infty} g_p\ H_{K+p-1}\left(\frac{u}{2Q}\right) \quad \text{for } u > 0 \ . \tag{D-35}$$

Alternatively, we have the more rapidly convergent form

$$\text{Prob}(\gamma > u) = 1 - F \sum_{p=0}^{\infty} g_p \left[1 - H_{K+p-1}\left(\frac{u}{2Q}\right)\right] \quad \text{for } u > 0 \ . \tag{D-36}$$

PROBABILITY DENSITY FROM CHARACTERISTIC FUNCTION (163)

The characteristic function of normalized random variable $\phi = q/R_{11}(0,0)$ defined in (161) was presented in (163). The corresponding probability density function $p_\phi(u)$ was then evaluated for a few special cases in (166), (171), (175), and (180). More generally, if we apply the expansion technique above to general result (163), we obtain the following procedure for $p_\phi(u)$: given M, $\{r^{(m)}\}$, $\{\eta_m^2\}$, and threshold u,

$$F = \left(\prod_{m=1}^{M} r^{(m)}\right)^{-\frac{1}{2}} , \qquad \alpha_o = -\frac{1}{2}\sum_{m=1}^{M} \eta_m^2 , \tag{D-37}$$

$$p\,\alpha_p = \frac{1}{2}\sum_{m=1}^{M} \frac{\left(r^{(m)} - 1\right)^{p-1}}{r^{(m)p}} \left(r^{(m)} - 1 + p\,\eta_m^2\right) \quad \text{for } p \ge 1 , \tag{D-38}$$

$$g_o = \exp(\alpha_o) , \qquad g_p = \frac{1}{p}\sum_{n=1}^{p} n\,\alpha_n\,g_{p-n} \quad \text{for } p \ge 1 , \tag{D-39}$$

$$h_o(u) = \frac{u^{\frac{1}{2}M-1}\exp(-u/2)}{2^{M/2}\,\Gamma(M/2)} ,$$

$$h_p(u) = h_{p-1}(u)\,\frac{u}{M - 2 + 2p} \quad \text{for } p \ge 1 , \tag{D-40}$$

$$p_\phi(u) = F\sum_{p=0}^{\infty} g_p\,h_p(u) \quad \text{for } u > 0 . \tag{D-41}$$

A function subprogram for this particular procedure is listed on the next page.

FUNCTION SUBPROGRAM FOR (D-37) - (D-41)

```
 10   DEF FNPdf(DOUBLE M,REAL U,Rs(*),Etasq(*))
 20   DOUBLE Ms,Ps        !  INTEGERS
 30   ALLOCATE Q(1:M),R(1:M),S(1:M),A(1:100),G(0:100)
 40   M2=M/2.
 50   M21=M2-1.
 60   U2=U/2.
 70   F=1.
 80   S=0.
 90   FOR Ms=1 TO M
100   Rs=Rs(Ms)
110   T=1./Rs
120   E=Etasq(Ms)
130   Q(Ms)=1.
140   R(Ms)=1.-T
150   S(Ms)=E*T
160   F=F*Rs
170   S=S+E
180   NEXT Ms
190   G(0)=EXP(-.5*S)
200   H=.5*U2^M21*EXP(-U2)/(SQR(F)*FNGamma(M2))
210   Pdf=G(0)*H
220   Q=1.
230   FOR Ps=1 TO 100
240   S=0.
250   FOR Ms=1 TO M
260   R=R(Ms)
270   IF Ps=1 THEN 290
280   Q(Ms)=Q=Q(Ms)*R
290   S=S+Q*(R+Ps*S(Ms))
300   NEXT Ms
310   A(Ps)=.5*S
320   S=0.
330   FOR Ms=1 TO Ps
340   S=S+A(Ms)*G(Ps-Ms)
350   NEXT Ms
360   G(Ps)=G=S/Ps
370   H=H*U2/(M21+Ps)
380   T=G*H
390   Pdf=Pdf+T
400   IF T<Pdf*1.E-15 THEN 440
410   NEXT Ps
420   PRINT "100 TERMS ARE INSUFICIENT"
430   PAUSE
440   RETURN Pdf
450   END
```

MAIN PROGRAM

A program for the evaluation of (D-21), by means of (D-10) and recursions (D-12) and (D-18), is presented below. The desired false alarm and detection probabilities, P_f and P_d respectively, of the processor in figure 1 are specified in lines 10 and 20. Parameter K in line 30 is the number of signal pulses, while M in line 40 is the number of components in fading model (45). The M power ratios $\{r^{(m)}\}$ are inputted in line 50, while the MK deterministic input signal-to-noise ratio measures $\{D_{km}/N_o\}$ are entered in line 60. The K time locations $\{t_k\}$ and frequency locations $\{f_k\}$ of the K transmitted signal pulses are entered in lines 70 and 80. Guesses at the required average random input signal-to-noise ratio measure E_1/N_o are required in lines 90 and 100; the program will search for the required threshold u and required input signal-to-noise ratio E_1/N_o to meet the specifications.

The normalized covariance function of the medium fading is described in function subprogram DEF FNCov(T,F); currently, it allows for exponential decay in time separation τ and frequency separation ν, but can be easily modified. It must be noted that covariance $R_{mn}(\tau,\nu)$ defined in (47) involves a product of the amplitude-fading quantities $\{g_m(t,f)\}$, not their squares. This distinction, relative to covariance $\tilde{R}_{kj}$ in (55) and normalized covariance ρ_{kj} in (56) between power-fading variates $\{g_m^2(t,f)\}$, must be carefully observed and maintained; furthermore, it is worthwhile to review (66) and (69) at this point, which pertain

only to special cases 5 and 7.

Quantities that need be computed only once and then used repeatedly for different signal-to-noise ratios, such as the eigenvalues and eigenvectors and some auxiliary arrays, are evaluated in the main program and passed in common to function subprogram DEF FNPd for the detection probability. Similarly, for speed and storage purposes, we have identified, for use in this subprogram, the array variables $A(p) = p\ \alpha_p$ for $p \geq 1$ and $G(p) = F\ g_p$ for $p \geq 0$. A separate listing for the false alarm probability P_F in (188) is given in function subprogram DEF FNPf.

The program is listed in BASIC for the Hewlett-Packard 9000 computer. On this particular device, the notation DOUBLE denotes INTEGER variables, not double precision. As a numerical check, the values printed out are threshold $U = 38.2583364$ and signal-to-noise ratio measure $dE = 10\ \log_{10}(E_1/N_o) = 13.4744256$. Since the mean of processor output γ for noise-only is $\overline{\gamma} = 2K = 6$, the normalized threshold is $U/\overline{\gamma} = 6.376$, for this example of $P_F = 1E\text{-}6$.

```
10    Pf=1.E-6                  !  FALSE ALARM PROBABILITY
20    Pd=.9                     !  DETECTION PROBABILITY
30    K=3                       !  NUMBER OF SIGNAL PULSES
40    M=2                       !  NUMBER OF FADING COMPONENTS
50    DATA 1.,1.                !  POWER RATIOS r(m)    FOR 1 <= m <= M
60    DATA 0,0,0,0,0,0          !  Dkm/No FOR 1 <= k <= K, 1 <= m <= M
70    DATA .1,.3,.5             !  TIMES tk (SEC)       FOR 1 <= k <= K
80    DATA .2,.6,.4             !  FREQUENCIES fk (HZ) FOR 1 <= k <= K
90    E1no0=10.                 !  E1/No STARTING VALUE
100   E1no1=1.                  !  E1/No INCREMENT
110   Ef=1.E-15                 !  TOLERANCE ON Pf
120   Ed=1.E-10                 !  TOLERANCE ON Pd
130   PRINT
140   PRINT "Pf =";Pf;"     K =";K;"     M =";M
150   OPTION BASE 1
160   COM DOUBLE K,M            !  INTEGERS (NOT DOUBLE PRECISION)
170   DOUBLE Ms,Ks,Js           !  INTEGERS
180   DIM Rs(5),Dn(10,5),Ts(10),Fs(10),Psi(5),Cbar(10,10)
190   DIM U(10,10),V(10,10),Eig(10),Sq(10,5)
200   COM U,Prod(5,10),Es(5,10),T(5,10),Q(5,10),S(5,10),B(5,10)
210   COM A(100),G(0:100)
220   REDIM Rs(1:M),Dn(1:K,1:M),Ts(1:K),Fs(1:K),Psi(1:M),Cbar(1:K,1:K)
230   REDIM U(1:K,1:K),V(1:K,1:K),Eig(1:K),Sq(1:K,1:M)
240   REDIM Prod(1:M,1:K),Es(1:M,1:K),T(1:M,1:K)
250   REDIM Q(1:M,1:K),S(1:M,1:K),B(1:M,1:K)
260   READ Rs(*),Dn(*),Ts(*),Ps(*)     !  Dn(*) WAS FILLED IN THE ORDER:
270   S=0.            !  Dn(1,1),Dn(1,2),Dn(1,3),...,Dn(k,m),...,Dn(K,M)
280   FOR Ms=1 TO M
290   S=S+Rs(Ms)
300   NEXT Ms
310   FOR Ms=1 TO M
320   Psi(Ms)=Rs(Ms)/S
330   NEXT Ms
340   FOR Ks=1 TO K
350   FOR Js=1 TO K
360   Cov=FNCov(Ts(Ks)-Ts(Js),Fs(Ks)-Fs(Js))
370   Cbar(Ks,Js)=Cov           !  NORMALIZED COVARIANCE MATRIX
380   NEXT Js
390   NEXT Ks
400   MAT U=Cbar
410   CALL Svd(K,K,U(*),V(*),Eig(*))   !  OUTPUTS: U(*),V(*),Eig(*)
420   PRINT "EIGENVALUES:"
430   PRINT Eig(*)
```

```
440   FOR Ms=1 TO M
450   T=2.*Psi(Ms)
460   FOR Ks=1 TO K
470   Prod(Ms,Ks)=T*Eig(Ks) !  2 Psi Eig
480   NEXT Ks
490   NEXT Ms
500   FOR Ks=1 TO K
510   FOR Ms=1 TO M
520   Sq(Ks,Ms)=SQR(2.*Dn(Ks,Ms))
530   NEXT Ms
540   NEXT Ks
550   FOR Ks=1 TO K
560   FOR Ms=1 TO M
570   S=0.
580   FOR Js=1 TO K
590   S=S+V(Js,Ks)*Sq(Js,Ms)
600   NEXT Js
610   Es(Ms,Ks)=S*S          !  emk
620   NEXT Ms
630   NEXT Ks
640   U1=1.                  !  THRESHOLD INCREMENT
650   U0=-LOG(Pf)-U1         !  THRESHOLD STARTING VALUE
660   CALL Inversfunction1(-Pf,Ef,U0,U1,U)
670   PRINT "THRESHOLD U =";U
680   CALL Inversfunction2(Pd,Ed,E1no0,E1no1,E1no)
690   Db=10.*LGT(E1no)
700   PRINT "dB =";Db
710   END
720   !
730   DEF FNCov(Tau,Nu)      !  NORMALIZED COVARIANCE
740   A=LOG(2.)*.5           !  R11(Tau,Nu)/R11(0,0)
750   B=1.3
760   Cov=EXP(-A*ABS(Tau)-B*ABS(Nu))
770   RETURN Cov
780   FNEND
790   !
800   DEF FNPf(U)            !  PROBABILITY OF FALSE ALARM
810   COM DOUBLE K           !  INTEGER
820   DOUBLE Ks
830   U2=U*.5
840   Pf=T*EXP(-U2)
850   FOR Ks=1 TO K-1
860   T=T*U2/Ks
870   Pf=Pf+T
880   NEXT Ks
890   RETURN -Pf             !  - TO YIELD INCREASING FUNCTION
900   FNEND
910   !
```

```
920   DEF FNPd(E1no)
930   COM DOUBLE K,M          !  INTEGERS
940   COM U,Prod(*),Es(*),T(*),Q(*),S(*),B(*),A(*),G(*)
950   DOUBLE Ms,Ks,K1,P
960   Tol=1.E-10              !  RELATIVE ERROR OF SUM
970   FOR Ms=1 TO M
980   FOR Ks=1 TO K
990   T(Ms,Ks)=E1no*Prod(Ms,Ks)
1000  NEXT Ks
1010  NEXT Ms
1020  F=1.
1030  S=0.
1040  FOR Ms=1 TO M
1050  FOR Ks=1 TO K
1060  T=T(Ms,Ks)
1070  Q(Ms,Ks)=Q=1.+T
1080  S(Ms,Ks)=T/Q
1090  F=F*Q
1100  S=S+Es(Ms,Ks)/Q
1110  NEXT Ks
1120  NEXT Ms
1130  G(0)=EXP(-.5*S)/SQR(F)
1140  K1=K-1
1150  U2=U*.5
1160  T=EXP(-U2)
1170  H1=1.-T
1180  FOR Ks=1 TO K1
1190  T=T*U2/Ks
1200  H1=H1-T
1210  NEXT Ks
1220  Pd=1.-G(0)*H1
1230  FOR P=1 TO 100
1240  T=T*U2/(K1+P)
1250  H1=H1-T
1260  S=0.
1270  FOR Ms=1 TO M
1280  FOR Ks=1 TO K
1290  Q=Q(Ms,Ks)
1300  IF P>1 THEN 1330
1310  B(Ms,Ks)=B=1./Q
1320  GOTO 1340
1330  B(Ms,Ks)=B=B(Ms,Ks)*S(Ms,Ks)
1340  S=S+B*(T(Ms,Ks)/P+Es(Ms,Ks)/Q)
1350  NEXT Ks
1360  NEXT Ms
1370  A(P)=.5*S*P
1380  S=0.
1390  FOR Ms=1 TO P
1400  S=S+A(Ms)*G(P-Ms)
1410  NEXT Ms
1420  G(P)=G=S/P
1430  Del=G*H1
1440  Pd=Pd-Del
1450  IF Del<Pd*Tol THEN 1490
1460  NEXT P
1470  PRINT "100 TERMS ARE INSUFFICIENT"
1480  PAUSE
1490  PRINT "Pd =";Pd,"    P =";P;"     E1/No =";E1no
1500  RETURN Pd
1510  FNEND
1520  !
```

```
1530    SUB Inversfunction1(Desired,Error,X1,Del,X2)
1540    X2=X1+Del
1550    F1=FNPf(X1)
1560    F2=FNPf(X2)
1570    IF F2>=Desired THEN 1620
1580    X1=X2
1590    X2=X2+Del
1600    F1=F2
1610    GOTO 1560
1620    IF F1<=Desired THEN 1680
1630    X2=X1
1640    X1=X1-Del
1650    F2=F1
1660    F1=FNPf(X1)
1670    GOTO 1620
1680    Xa=X1
1690    Xb=X2
1700    IF F2-Desired<Desired-F1 THEN 1770
1710    T=X1
1720    X1=X2
1730    X2=T
1740    T=F1
1750    F1=F2
1760    F2=T
1770    IF ABS(F2-Desired)<Error THEN 1870
1780    IF F2=F1 THEN 1870
1790    T=(X1*(F2-Desired)-X2*(F1-Desired))/(F2-F1)
1800    T=MAX(T,Xa)
1810    T=MIN(T,Xb)
1820    X1=X2
1830    X2=T
1840    F1=F2
1850    F2=FNPf(X2)
1860    GOTO 1770
1870    SUBEND
1880    !
1890    SUB Inversfunction2(Desired,Error,X1,Del,X2)
1900    X2=X1+Del
1910    F1=FNPd(X1)
1920    F2=FNPd(X2)
1930    IF F2>=Desired THEN 1980
1940    X1=X2
1950    X2=X2+Del
1960    F1=F2
1970    GOTO 1920
1980    IF F1<=Desired THEN 2040
1990    X2=X1
2000    X1=X1-Del
```

```
2010    F2=F1
2020    F1=FNPd(X1)
2030    GOTO 1980
2040    Xa=X1
2050    Xb=X2
2060    IF F2-Desired<Desired-F1 THEN 2130
2070    T=X1
2080    X1=X2
2090    X2=T
2100    T=F1
2110    F1=F2
2120    F2=T
2130    IF ABS(F2-Desired)<Error THEN 2230
2140    IF F2=F1 THEN 2230
2150    T=(X1*(F2-Desired)-X2*(F1-Desired))/(F2-F1)
2160    T=MAX(T,Xa)
2170    T=MIN(T,Xb)
2180    X1=X2
2190    X2=T
2200    F1=F2
2210    F2=FNPd(X2)
2220    GOTO 2130
2230    SUBEND
2240    !
2250    SUB Svd(DOUBLE M,N,REAL A(*),V(*),W(*))
2260  ! THIS SUBROUTINE COMPUTES THE SINGULAR VALUE DECOMPOSITION
2270  ! OF AN ARBITRARY REAL MxN MATRIX A:  A = U W Vt, M >= N.
2280  ! U IS MxN, V IS NxN, W IS NxN:  W = DIAG(D(n)).
2290    ALLOCATE Rv1(1:N)              !  NUMERICAL RECIPES, PAGES 60-64
2300    IF M>=N THEN 2330              !  A(*) IS OVER-WRITTEN
2310    PRINT "M<N IS DISALLOWED"
2320    PAUSE
2330    DOUBLE I,J,K,L,Its,Nm,Jj       !  INTEGERS (NOT DOUBLE PRECISION)
2340    G=Scale=Anorm=0.
2350    FOR I=1 TO N
2360    L=I+1
2370    Rv1(I)=Scale*G
2380    G=S=Scale=0.
2390    IF I>M THEN 2670
2400    FOR K=I TO M
2410    Scale=Scale+ABS(A(K,I))
2420    NEXT K
2430    IF Scale=0. THEN 2670
2440    FOR K=I TO M
2450    Aa=A(K,I)=A(K,I)/Scale
2460    S=S+Aa*Aa
2470    NEXT K
2480    F=A(I,I)
2490    G=-SQR(S)
2500    IF F<0. THEN G=-G
```

```
2510    H=F*G-S
2520    A(I,I)=F-G
2530    IF I=N THEN 2640
2540    FOR J=L TO N
2550    S=0.
2560    FOR K=I TO M
2570    S=S+A(K,I)*A(K,J)
2580    NEXT K
2590    F=S/H
2600    FOR K=I TO M
2610    A(K,J)=A(K,J)+F*A(K,I)
2620    NEXT K
2630    NEXT J
2640    FOR K=I TO M
2650    A(K,I)=A(K,I)*Scale
2660    NEXT K
2670    W(I)=Scale*G
2680    G=S=Scale=0.
2690    IF (I>M) OR (I=N) THEN 2990
2700    FOR K=L TO N
2710    Scale=Scale+ABS(A(I,K))
2720    NEXT K
2730    IF Scale=0. THEN 2990
2740    FOR K=L TO N
2750    Aa=A(I,K)=A(I,K)/Scale
2760    S=S+Aa*Aa
2770    NEXT K
2780    F=A(I,L)
2790    G=-SQR(S)
2800    IF F<0. THEN G=-G
2810    H=F*G-S
2820    A(I,L)=F-G
2830    FOR K=L TO N
2840    Rv1(K)=A(I,K)/H
2850    NEXT K
2860    IF I=M THEN 2960
2870    FOR J=L TO M
2880    S=0.
2890    FOR K=L TO N
2900    S=S+A(J,K)*A(I,K)
2910    NEXT K
2920    FOR K=L TO N
2930    A(J,K)=A(J,K)+S*Rv1(K)
2940    NEXT K
2950    NEXT J
2960    FOR K=L TO N
2970    A(I,K)=A(I,K)*Scale
2980    NEXT K
2990    Anorm=MAX(Anorm,ABS(W(I))+ABS(Rv1(I)))
3000    NEXT I
```

```
3010    FOR I=N TO 1 STEP -1
3020    IF I>=N THEN 3190
3030    IF G=0. THEN 3160
3040    FOR J=L TO N
3050    V(J,I)=A(I,J)/A(I,L)/G
3060    NEXT J
3070    FOR J=L TO N
3080    S=0.
3090    FOR K=L TO N
3100    S=S+A(I,K)*V(K,J)
3110    NEXT K
3120    FOR K=L TO N
3130    V(K,J)=V(K,J)+S*V(K,I)
3140    NEXT K
3150    NEXT J
3160    FOR J=L TO N
3170    V(I,J)=V(J,I)=0.
3180    NEXT J
3190    V(I,I)=1.
3200    G=Rv1(I)
3210    L=I
3220    NEXT I
3230    FOR I=N TO 1 STEP -1
3240    L=I+1
3250    G=W(I)
3260    IF I>=N THEN 3300
3270    FOR J=L TO N
3280    A(I,J)=0.
3290    NEXT J
3300    IF G=0. THEN 3470
3310    G=1./G
3320    IF I=N THEN 3430
3330    FOR J=L TO N
3340    S=0.
3350    FOR K=L TO M
3360    S=S+A(K,I)*A(K,J)
3370    NEXT K
3380    F=S/A(I,I)*G
3390    FOR K=I TO M
3400    A(K,J)=A(K,J)+F*A(K,I)
3410    NEXT K
3420    NEXT J
3430    FOR J=I TO M
3440    A(J,I)=A(J,I)*G
3450    NEXT J
3460    GOTO 3500
3470    FOR J=I TO M
3480    A(J,I)=0.
3490    NEXT J
3500    A(I,I)=A(I,I)+1.
```

```
3510    NEXT I
3520    FOR K=N TO 1 STEP -1
3530    FOR Its=1 TO 30
3540    FOR L=K TO 1 STEP -1
3550    Nm=L-1
3560    IF (ABS(Rv1(L))+Anorm)=Anorm THEN 3780
3570    IF (ABS(W(Nm))+Anorm)=Anorm THEN 3590
3580    NEXT L
3590    C=0.
3600    S=1.
3610    FOR I=L TO K
3620    F=S*Rv1(I)
3630    Rv1(I)=C*Rv1(I)
3640    IF (ABS(F)+Anorm)=Anorm THEN 3780
3650    G=W(I)
3660    H=SQR(F*F+G*G)
3670    W(I)=H
3680    H=1./H
3690    C=G*H
3700    S=-F*H
3710    FOR J=1 TO M
3720    Y=A(J,Nm)
3730    Z=A(J,I)
3740    A(J,Nm)=Y*C+Z*S
3750    A(J,I)=-Y*S+Z*C
3760    NEXT J
3770    NEXT I
3780    Z=W(K)
3790    IF L<>K THEN 3860
3800    IF Z>=0. THEN 3850
3810    W(K)=-Z
3820    FOR J=1 TO N
3830    V(J,K)=-V(J,K)
3840    NEXT J
3850    GOTO 4390
3860    IF Its<30 THEN 3890
3870    PRINT "NO CONVERGENCE IN 30 ITERATIONS"
3880    PAUSE
3890    X=W(L)
3900    Nm=K-1
3910    Y=W(Nm)
3920    G=Rv1(Nm)
3930    H=Rv1(K)
3940    F=((Y-Z)*(Y+Z)+(G-H)*(G+H))/(2.*H*Y)
3950    G=SQR(F*F+1.)
3960    Aa=ABS(G)
3970    IF F<0. THEN Aa=-Aa
3980    F=((X-Z)*(X+Z)+H*((Y/(F+Aa))-H))/X
3990    C=S=1.
4000    FOR J=L TO Nm
```

```
4010    I=J+1
4020    G=Rv1(I)
4030    Y=W(I)
4040    H=S*G
4050    G=C*G
4060    Z=SQR(F*F+H*H)
4070    Rv1(J)=Z
4080    C=F/Z
4090    S=H/Z
4100    F=X*C+G*S
4110    G=-X*S+G*C
4120    H=Y*S
4130    Y=Y*C
4140    FOR Jj=1 TO N
4150    X=V(Jj,J)
4160    Z=V(Jj,I)
4170    V(Jj,J)=X*C+Z*S
4180    V(Jj,I)=-X*S+Z*C
4190    NEXT Jj
4200    Z=SQR(F*F+H*H)
4210    W(J)=Z
4220    IF Z=0. THEN 4260
4230    Z=1./Z
4240    C=F*Z
4250    S=H*Z
4260    F=C*G+S*Y
4270    X=-S*G+C*Y
4280    FOR Jj=1 TO M
4290    Y=A(Jj,J)
4300    Z=A(Jj,I)
4310    A(Jj,J)=Y*C+Z*S
4320    A(Jj,I)=-Y*S+Z*C
4330    NEXT Jj
4340    NEXT J
4350    Rv1(L)=0.
4360    Rv1(K)=F
4370    W(K)=X
4380    NEXT Its
4390    NEXT K
4400    SUBEND
```

APPENDIX E. SOLUTION OF GENERALIZED EIGENVALUE PROBLEM

C and B are N×N real symmetric matrices, and B is positive definite. The important alternative case where B is not positive definite will be undertaken in (E-14) and sequel. We want the normalized modal matrix Q and the eigenvalue matrix Λ of the generalized characteristic-value problem [18; page 74] encountered in (B-10), namely

$$C\ Q = B^{-1}\ Q\ \Lambda\ . \tag{E-1}$$

Then, it is known that [18; pages 74 - 77]

$$Q^T\ B^{-1}\ Q = I \quad \text{and} \quad Q^T\ C\ Q = \Lambda\ , \tag{E-2}$$

which is a simultaneous reduction of two matrices to diagonal form. Alternatively, when both equations in (E-2) are satisfied, then (E-1) follows.

The first relation in (E-2) sets the scalings on the eigenvectors $\{V_n\}$ of Q; in fact, it reads $V_n^T\ B^{-1}\ V_n = 1$ for $1 \le n \le N$. An alternative scaling choice would be to make the eigenvectors of unit length, that is, $V_n^T\ V_n = 1$. However, this would result in $Q^T\ B^{-1}\ Q$ becoming a diagonal matrix with non-unity elements; this latter alternative is not adopted here. Therefore, the eigenvectors $\{V_n\}$ in (E-2) do not have unit length, that is, $V_n^T\ V_n \ne 1$ generally. The eigenvectors are unique within their polarities.

A procedure for determining Q and Λ is now presented. Solve for the square root matrix S in

$$B = S\ S^T\ , \tag{E-3}$$

where S need not be symmetric. For example, the lower (or upper) triangular square root matrix will suffice. Then there follows

$$B^{-1} = S^{-T} S^{-1} \quad \text{and} \quad S^T B^{-1} S = I \; . \tag{E-4}$$

Now compute N×N matrix

$$A = S^T C S \; , \tag{E-5}$$

which is symmetric. Solve the <u>standard</u> eigenvalue problem

$$A U = U \gamma \tag{E-6}$$

for normalized modal matrix U and diagonal eigenvalue matrix γ. Then it is known that these solutions satisfy

$$U^T U = I \quad \text{and} \quad U^T A U = \gamma \; . \tag{E-7}$$

It will now be shown that

$$Q = S U \quad \text{and} \quad \Lambda = \gamma \tag{E-8}$$

are the desired matrix solutions to (E-1). Substitution of (E-8) in the first relation in (E-2) yields

$$Q^T B^{-1} Q = U^T S^T B^{-1} S U = U^T I U = I \; , \tag{E-9}$$

where we used (E-4) and (E-7). Also, substitution of (E-8) in the second relation in (E-2) yields

$$Q^T C Q - \Lambda = U^T S^T C S U - \gamma = U^T A U - \gamma = 0 \; , \tag{E-10}$$

where we used (E-5) and (E-7). Thus, both relations in (E-2) are satisfied by the matrix assignments in (E-8).

In summary, in order to solve (E-1): compute square root matrix S in (E-3); evaluate matrix A via (E-5); solve (E-6) for U and γ; determine Q and Λ by means of (E-8). Notice there is never any need to compute inverse matrix B^{-1}; however, matrix B must be positive definite for the operation in (E-3) to succeed.

We now present an example which illustrates many of the quantities considered above; let matrices

$$B = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}. \tag{E-11}$$

Then the solutions of (E-1) are

$$\Lambda = \begin{bmatrix} 20 & 0 \\ 0 & 12 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}. \tag{E-12}$$

The column vectors of Q are not orthogonal; in fact,

$$Q^T Q = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}. \tag{E-13}$$

Rather, the orthogonality relations of (E-2) may be readily verified to be satisfied in this example, namely $Q^T B^{-1} Q = I$ and $Q^T C Q = \Lambda$.

GENERAL SYMMETRIC MATRIX B

In attempting to solve (E-1) for Q and Λ, the square root of matrix B was required in (E-3). If B is not positive definite, complex solutions would be required there. Here, we will solve the alternative generalized characteristic-value matrix equation encountered in (B-44), namely

$$B\,Q = C^{-1}\,Q\,\Lambda\,, \tag{E-14}$$

where covariance matrix C is real symmetric and positive definite, while matrix B is real symmetric but can be indefinite. The solutions Q and Λ will then satisfy the relations

$$Q^T\,C^{-1}\,Q = I \quad \text{and} \quad Q^T\,B\,Q = \Lambda\,. \tag{E-15}$$

To this aim, first solve for the square root matrix S in

$$C = S\,S^T\,. \tag{E-16}$$

Then

$$C^{-1} = S^{-T}\,S^{-1} \quad \text{and} \quad S^T\,C^{-1}\,S = I\,. \tag{E-17}$$

Next, compute the N×N matrix

$$A = S^T\,B\,S\,, \tag{E-18}$$

which is symmetric. Solve the standard characteristic-value equation

$$\mathbf{A}\,\mathbf{U} = \mathbf{U}\,\gamma \tag{E-19}$$

for U and γ, for which we have orthogonality properties

$$U^T U = I \quad \text{and} \quad U^T A U = \gamma . \tag{E-20}$$

It will now be shown that the solutions to (E-14) are

$$Q = S U \quad \text{and} \quad \Lambda = \gamma . \tag{E-21}$$

Substitution of (E-21) in the first relation in (E-15) yields

$$Q^T C^{-1} Q = U^T S^T C^{-1} S U = U^T I U = U^T U = I , \tag{E-22}$$

where we used (E-17) and (E-20). Similarly, substitution of (E-21) in the second relation in (E-15) yields

$$Q^T B Q - \Lambda = U^T S^T B S U - \gamma = U^T A U - \gamma = 0 , \tag{E-23}$$

where we used (E-18) and (E-20). Thus, both relations in (E-15) are verified by solutions (E-21).

In summary, in order to solve (E-14): compute square root matrix S in (E-16); evaluate matrix A according to (E-18); solve (E-19) for U and γ; determine Q and Λ by means of (E-21). This procedure requires covariance matrix C to be positive definite. Inverses of matrices C, B, or S are not required.

(If square root matrix S in (E-16) is taken lower triangular, then S^{-1} is also lower triangular and can be very easily computed directly from S. This enables calculation of inverse matrix C^{-1} by means of (E-17), if desired.)

INTERRELATIONSHIPS OF SOLUTIONS

Here we shall relate the solutions Q and Λ of (E-1) to the solutions Q and Λ of (E-14). In particular, we maintain that the specific relations are

$$Q = Q^{-T} \Lambda^{\frac{1}{2}} \quad \text{and} \quad \Lambda = \Lambda \; . \tag{E-24}$$

To verify this claim, substitute (E-24) into the first relation in (E-2) and obtain

$$Q^T B^{-1} Q - I = \Lambda^{\frac{1}{2}} Q^{-1} B^{-1} Q^{-T} \Lambda^{\frac{1}{2}} - I = \Lambda^{\frac{1}{2}} \Lambda^{-1} \Lambda^{\frac{1}{2}} \quad I = 0 \; , \tag{E-25}$$

where we used (E-15). Similarly, substitution in the second relation in (E-2) yields

$$Q^T C Q - \Lambda = \Lambda^{\frac{1}{2}} Q^{-1} C Q^{-T} \Lambda^{\frac{1}{2}} - \Lambda = \Lambda^{\frac{1}{2}} I \Lambda^{\frac{1}{2}} - \Lambda = 0 \; , \tag{E-26}$$

where we also used (E-15). Thus, both relations in (E-2) are satisfied by interrelationships (E-24).

The connection between eigenvectors is obtained by utilizing (E-15) to give

$$Q^{-T} = B Q \Lambda^{-1} \; . \tag{E-27}$$

Substitution in (E-24) then yields

$$Q = B Q \Lambda^{-\frac{1}{2}} \; ; \quad \text{that is,} \quad V_n = B V_n \lambda_n^{-\frac{1}{2}} \; . \tag{E-28}$$

The eigenvalue relation is, directly from (E-24),

$$\lambda_n = \lambda_n \; . \tag{E-29}$$

Finally, the connection between vectors $\underline{E}$ and $\underline{A}$ in (B-18) and the corresponding vectors $\underline{E}$ and $\underline{A}$ in (B-51) and (B-52) is as follows: the use of (E-28) in the first relation in (B-18) yields

$$\varepsilon_n = V_n^T E = \lambda_n^{-\frac{1}{2}} V_n^T B E = \lambda_n^{\frac{1}{2}} \varepsilon_n , \tag{E-30}$$

the last relation based on (B-59). And substitution of (E-28) in the second relation in (B-18) yields

$$\alpha_n = V_n^T B^{-1} A = \lambda_n^{-\frac{1}{2}} V_n^T B B^{-1} A = \lambda_n^{-\frac{1}{2}} \alpha_n , \tag{E-31}$$

using (B-60).

It can now be seen that the use of (E-29) - (E-31) in characteristic function (B-20) immediately converts it to (B-54). This is a direct confirmation of the fact that the characteristic function of q is unique, regardless of how derived. However, the recommended procedure is that given in (B-43) and sequel, because it is more general, allowing symmetric matrix B to be indefinite. In fact, relation (E-28) is valid only when <u>all</u> eigenvalues λ_n are <u>positive</u>. If this is not the case, although Q cannot be found using real arithmetic, the technique summarized under (E-23), for determining solutions Q and Λ of (E-14), is still valid and can <u>always</u> be used with real arithmetic. Eigenvalues $\{\lambda_n\}$ can take on positive and negative values. The only restriction is that symmetric matrix C be positive definite.

APPENDIX F. STATISTICS OF REAL BILINEAR FORM

Suppose we encounter real bilinear form [22; (13.5-1)]

$$b = X^T A Y , \tag{F-1}$$

where random vector X is M×1, random vector Y is N×1, and arbitrary matrix A is M×N, all quantities being real. We allow random vectors X and Y to be arbitrarily correlated with each other, but to have zero means here; generalizations to nonzero means can be made along the lines of appendix B.

Let L = M+N and define L×1 vector

$$Z = \begin{bmatrix} X \\ Y \end{bmatrix} . \tag{F-2}$$

Let L×L matrix

$$B = \frac{1}{2} \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} = B^T . \tag{F-3}$$

Then (F-1) becomes

$$b = Z^T B Z , \tag{F-4}$$

which is a quadratic form in L variables; L = M+N. Matrix B is symmetric but not positive definite. Letting r = min(M,N), numerical investigation has revealed that B has r positive eigenvalues $\mu_1,\ldots,\mu_r$; r eigenvalues which are their negatives $-\mu_1,\ldots,-\mu_r$; and the rest zero.

The mean of random vector Z is zero, and its (arbitrary) L×L covariance matrix is

$$C = \overline{Z\ Z^T} = \overline{\begin{bmatrix} X \\ Y \end{bmatrix} [X^T \quad Y^T]} = \begin{bmatrix} \overline{X\ X^T} & \overline{X\ Y^T} \\ \overline{Y\ X^T} & \overline{Y\ Y^T} \end{bmatrix} = \begin{bmatrix} C_{xx} & C_{xy} \\ C_{xy}^T & C_{yy} \end{bmatrix} , \qquad \text{(F-5)}$$

which is symmetric. C is also nonnegative definite, because

$$V^T\ C\ V = V^T\ \overline{Z\ Z^T}\ V = \overline{\left(V^T\ Z\right)^2} \geq 0 \qquad \text{(F-6)}$$

for any $L \times 1$ vector V. We presume that C is positive definite.

In order to convert quadratic form (F-4) to diagonal form with uncorrelated variates, we first solve the $L \times L$ generalized characteristic-value equation (see (E-14) and sequel)

$$B\ Q = C^{-1}\ Q\ \Lambda \qquad \text{(F-7)}$$

for $L \times L$ matrices Q and Λ. Then, we have properties [18; pages 74 - 77]

$$Q^T\ C^{-1}\ Q = I \quad \text{and} \quad Q^T\ B\ Q = \Lambda = \operatorname{diag}[\lambda_1\ .\ .\ .\ \lambda_L]\ . \qquad \text{(F-8)}$$

Now let linearly transformed random vector

$$W = Q^{-1}\ Z = [w_1\ .\ .\ .\ w_L] \quad ; \qquad Z = Q\ W\ . \qquad \text{(F-9)}$$

Vector W also has zero mean. Its covariance matrix is

$$\overline{W\ W^T} = Q^{-1}\ \overline{Z\ Z^T}\ Q^{-T} = Q^{-1}\ C\ Q^{-T} = I\ , \qquad \text{(F-10)}$$

upon use of (F-5) and (F-8). Also, bilinear form (F-4) becomes

$$b = Z^T\ B\ Z = W^T\ Q^T\ B\ Q\ W = W^T\ \Lambda\ W = \sum_{k=1}^{L} \lambda_k w_k^2\ , \qquad \text{(F-11)}$$

where we used (F-9) and (F-8). Thus, b is a sum of L weighted squares of uncorrelated zero-mean unit-variance random variables. Notice that not all eigenvalues $\{\lambda_k\}$ need be positive; some can be zero and some can be negative.

If random vectors X and Y are joint-Gaussian, then vector W in (F-9) is also Gaussian, with probability density function

$$p(W) = \prod_{k=1}^{L}\left((2\pi)^{-\frac{1}{2}} \exp(-w_k^2/2)\right) , \tag{F-12}$$

based on (F-10). The characteristic function of bilinear form b in (F-11) is then

$$f_b(\xi) = \overline{\exp(i\xi b)} = \prod_{k=1}^{L}\left(\int dw_k \; (2\pi)^{-\frac{1}{2}} \exp\left[- w_k^2/2 + i\xi\lambda_k w_k^2\right]\right) =$$

$$= \left[\prod_{k=1}^{L}\left(1 - i2\xi\lambda_k\right)\right]^{-\frac{1}{2}} . \tag{F-13}$$

Since only the eigenvalues $\{\lambda_k\}$ of Λ in (F-8) are required to evaluate characteristic function (F-13), it is not actually necessary to determine the eigenvectors Q in matrix equation (F-7). In fact, by a procedure identical to that given earlier in (B-55) - (B-58), it may be shown that $\{\lambda_k\}$ are the eigenvalues of nonsymmetric L×L matrix B C. Reference to (F-3) and (F-5) reveals that this latter matrix is given by

$$B\ C = \frac{1}{2} \begin{bmatrix} A\ C_{xy}^T & A\ C_{yy} \\ A^T\ C_{xx} & A^T\ C_{xy} \end{bmatrix} . \tag{F-14}$$

The rank of matrix B C is generally $R = 2 \min(M,N)$; that is, the product for characteristic function $f_b(\xi)$ in (F-13) will contain only R terms, which is less than L except when $M = N$.

The claim for the rank of B C is based upon the following observation. Suppose that $M \le N$; then (F-1) can be written as

$$b = \sum_{m=1}^{M} \sum_{n=1}^{N} x_m \, a_{mn} \, y_n = \sum_{m=1}^{M} x_m \, v_m \, , \qquad \text{(F-15)}$$

where (linearly transformed) Gaussian random variables

$$v_m = \sum_{n=1}^{N} a_{mn} \, y_n \quad \text{for } 1 \le m \le M \, . \qquad \text{(F-16)}$$

Thus, b is a sum of products of just 2M correlated random variables. Therefore, the characteristic function in (F-13) can have no more than 2M terms. Direct numerical evaluation of the eigenvalues $\{\lambda_k\}$ of nonsymmetric L×L matrix B C in (F-14) has verified that its rank is generally $R = 2 \min(M,N)$, and that the R nonzero eigenvalues can be positive and negative with no obvious interrelations. (If $N < M$, the summation order in (F-15) can be reversed, thereby giving b as a sum of products of 2N correlated random variables, and hence, rank 2N for matrix B C.)

The generalization of (F-1) to the form

$$b = X^T A_{11} X + X^T A_{12} Y + Y^T A_{21} X + Y^T A_{22} Y \qquad \text{(F-17)}$$

can be directly fit into the framework of (F-2) and (F-4) if we generalize definition (F-3) to L×L matrix

$$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} . \tag{F-18}$$

We can then replace this B matrix by its symmetric part, since only the symmetric part of B is active in quadratic form (F-4).

APPENDIX G. CHARACTERISTIC FUNCTION OF MOST GENERAL COMPLEX FORM WITH FIRST-ORDER AND SECOND-ORDER TERMS

Let Z be a complex N×1 random vector with real and imaginary parts X and Y; that is, Z = X + i Y. Let D_1 and D_2 be complex N×1 constant vectors. Let C_1, C_2, C_3, C_4 be complex N×N constant matrices, which need not be Hermitian or symmetric.

The most general first-order complex form is

$$f_1 = D_1^T Z + D_2^T Z^* = (D_1 + D_2)^T X + i(D_1 - D_2)^T Y = H^T W , \quad \text{(G-1)}$$

where

$$H = \begin{bmatrix} D_1 + D_2 \\ i(D_1 - D_2) \end{bmatrix} , \qquad W = \begin{bmatrix} X \\ Y \end{bmatrix} . \quad \text{(G-2)}$$

H is a complex 2N×1 constant vector and is completely arbitrary; that is, every complex element of H can be independently specified. W is a real 2N×1 random vector.

The most general second-order complex form is

$$f_2 = Z^H C_1 Z + Z^H C_2 Z^* + Z^T C_3 Z + Z^T C_4 Z^* = W^T M W , \quad \text{(G-3)}$$

where

$$M = \begin{bmatrix} C_1 + C_2 + C_3 + C_4 & i(C_1 - C_2 + C_3 - C_4) \\ -i(C_1 + C_2 - C_3 - C_4) & C_1 - C_2 - C_3 + C_4 \end{bmatrix} . \quad \text{(G-4)}$$

M is a complex 2N×2N constant matrix, which need not be Hermitian. M is completely arbitrary; that is, every complex element of M can be independently specified.

The pertinent statistics of real vector W are

$$\overline{W} = \begin{bmatrix} \overline{X} \\ \overline{Y} \end{bmatrix} , \quad K = \mathrm{Cov}(W) = \overline{(W - \overline{W})\ (W - \overline{W})^T} =$$

$$= \overline{\begin{bmatrix} X - \overline{X} \\ Y - \overline{Y} \end{bmatrix} \begin{bmatrix} X^T - \overline{X}^T & Y^T - \overline{Y}^T \end{bmatrix}} = \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} . \qquad \text{(G-5)}$$

Here, $\overline{W}$ is a real 2N×1 constant vector, while K is a real 2N×2N symmetric constant matrix.

The general <u>complex</u> form of interest is

$$c = f_2 + f_1 = W^T\, M\, W + H^T\, W . \qquad \text{(G-6)}$$

For X and Y joint-Gaussian, W is Gaussian with probability density function

$$p(W) = (2\pi)^{-N}\ (\det K)^{-\frac{1}{2}} \exp\left(- \frac{1}{2}(W - \overline{W})^T\, K^{-1}\, (W - \overline{W})\right) . \qquad \text{(G-7)}$$

The statistical quantity of interest is, for complex scalar α, the characteristic function of complex random variable c in (G-6); in particular, the characteristic function is defined here as the average of the exponential

$$f_c(\alpha) = \overline{\exp(\alpha c)} = \overline{\exp\left(\alpha\, W^T\, M\, W + \alpha\, H^T\, W\right)} = (2\pi)^{-N}\ (\det K)^{-\frac{1}{2}} \times$$

$$\times \int dW \exp\left(- \frac{1}{2}(W - \overline{W})^T\, K^{-1}\, (W - \overline{W}) + \alpha\, W^T\, M\, W + \alpha\, H^T\, W\right) =$$

$$= d^{-\frac{1}{2}} \exp(t) , \qquad \text{(G-8)}$$

where

$$d = \det(I - 2\,\alpha\,M\,K) = \det(I - 2\,\alpha\,K\,M)\ ,$$

$$t = -\frac{1}{2}\,\overline{W}^T\,K^{-1}\,\overline{W} + \frac{1}{2}\,V^T\,(I - 2\,\alpha\,K\,M)^{-1}\,K\,V\ ,$$

$$V = K^{-1}\,\overline{W} + \alpha\,H\ . \tag{G-9}$$

K is a real 2N×2N matrix, M is a complex 2N×2N matrix which need not be Hermitian, $\overline{W}$ is a real 2N×1 vector, H and V are complex 2N×1 vectors, and α is a complex scalar.

In general, we must invert 2N×2N real symmetric matrix K. Also, we must invert 2N×2N complex matrix $I - 2\,\alpha\,K\,M$, which need not be Hermitian, and which depends on argument α. The average of the exponential in (G-8) is the type of operation encountered in appendix C.

If $\overline{X} = 0$, $\overline{Y} = 0$, $D_1 = 0$, $D_2 = 0$, then $\overline{W} = 0$, $H = 0$, $V = 0$, $t = 0$, and we need only evaluate complex determinant $d = \det(I - 2\,\alpha\,K\,M)$, which depends on α.

SPECIAL CASE: $Y = 0$

Then complex forms

$$f_1 = (D_1 + D_2)^T X \equiv D^T X ,$$

$$f_2 = X^T (C_1 + C_2 + C_3 + C_4) X \equiv X^T C X ,$$

$$c = X^T C X + D^T X . \tag{G-10}$$

Matrices C and D are complex and completely arbitrary.

Identify in the subsection above,

$$2N \rightarrow N , \quad W \rightarrow X , \quad M \rightarrow C , \quad H \rightarrow D , \quad K \rightarrow K_{xx} , \tag{G-11}$$

thereby getting characteristic function

$$f_c(\alpha) = \overline{\exp(\alpha c)} = d^{-\frac{1}{2}} \exp(t) , \tag{G-12}$$

where

$$d = \det(I - 2 \alpha K_{xx} C) ,$$

$$t = - \frac{1}{2} \bar{X}^T K_{xx}^{-1} \bar{X} + \frac{1}{2} V^T (I - 2 \alpha K_{xx} C)^{-1} K_{xx} V ,$$

$$V = K_{xx}^{-1} \bar{X} + \alpha D . \tag{G-13}$$

K_{xx} is a real N×N matrix, C is a complex N×N matrix which need not be Hermitian, $\bar{X}$ is a real N×1 vector, D and V are complex N×1 vectors, and α is a complex scalar.

JOINT CHARACTERISTIC FUNCTION OF REAL AND IMAGINARY PARTS OF c

For the general complex form c in (G-6), the joint characteristic function of c_r and c_i is, for real ξ and ζ,

$$f(\xi,\zeta) \equiv \overline{\exp(i\xi\, c_r + i\zeta\, c_i)} \ . \tag{G-14}$$

But

$$\xi\, c_r + \zeta\, c_i = \xi\left(W^T M_r W + H_r^T W\right) + \zeta\left(W^T M_i W + H_i^T W\right) =$$

$$= W^T \tilde{M} W + \tilde{H}^T W \ , \tag{G-15}$$

where real matrices

$$\tilde{M} \equiv \xi\, M_r + \zeta\, M_i \ , \quad \tilde{H} \equiv \xi\, H_r + \zeta\, H_i \ . \tag{G-16}$$

Therefore, identifying $\alpha \to i$, $M \to \tilde{M}$, $H \to \tilde{H}$, in (G-8), there follows the joint characteristic function of c_r and c_i as

$$f(\xi,\zeta) = \overline{\exp\left(i\, W^T \tilde{M} W + i\, \tilde{H}^T W\right)} = d^{-\frac{1}{2}} \exp(t) \ , \tag{G-17}$$

where

$$d = \det(I - i\, 2\, \tilde{M} K) = \det(I - i\, 2\, K \tilde{M}) \ ,$$

$$t = -\frac{1}{2}\, \overline{W}^T K^{-1} \overline{W} + \frac{1}{2}\, V^T (I - i\, 2\, K \tilde{M})^{-1} K V \ ,$$

$$V = K^{-1} \overline{W} + i\, \tilde{H} \ . \tag{G-18}$$

REFERENCES

[1] J. I. Marcum, A Statistical Theory of Target Detection by Pulsed Radar: Mathematical Appendix, **IRE Transactions on Information Theory**, volume IT-6, number 2, pages 145 - 267, April 1960. Originally presented in Research Memorandum RM-753, RAND Corporation, Santa Monica, CA, 1 July 1948.

[2] P. Swerling, "Probability of Detection of Fluctuating Targets", **IRE Transactions on Information Theory**, volume IT-6, number 2, pages 269 - 308, April 1960. Originally presented in Research Memorandum RM-1217, RAND Corporation, Santa Monica, CA, 17 March 1954.

[3] M. Schwartz, "Effects of Signal Fluctuations in the Detection of Pulsed Signals in Noise", **IRE Transactions on Information Theory**, volume IT-2, number 2, pages 66 - 71, June 1956.

[4] P. Swerling, "Detection of Fluctuating Pulsed Signals in the Presence of Noise", **IRE Transactions on Information Theory**, volume IT-3, number 3, pages 175 - 178, September 1957.

[5] M. Nakagami, The m-Distribution - A General Formula of Intensity Distribution of Rapid Fading, **Statistical Methods in Radio Wave Propagation Proceedings**, UCLA, W. C. Hoffman, editor, Pergamon Press, 1958.

[6] P. Swerling, "Detection of Radar Echoes in Noise Revisited", **IEEE Transactions on Information Theory**, volume IT-12, number 3, pages 348 - 361, July 1966.

[7] M. Schwartz, W. R. Bennett, and S. Stein, **Communication Systems and Techniques**, McGraw-Hill Book Company, Inc., New York, NY, 1966.

[8] G. R. Heidbreder and R. L. Mitchell, "Detection Probabilities for Log-Normally Distributed Signals", **IEEE Transactions on Aerospace and Electronic Systems**, volume AES-3, number 1, pages 5 - 13, January 1967.

[9] R. A. Scholtz, J. J. Kappl, and N. E. Nahi, "The Detection of Moderately-Fluctuating Rayleigh Targets", **IEEE Transactions on Aerospace and Electronic Systems**, volume AES-12, number 2, pages 117 - 126, March 1976.

[10] A. H. Nuttall, **Operating Characteristics for Detection of a Fading Signal with K Dependent Fades and D-fold Diversity in M Alternative Locations**, NUSC Technical Report 5739, Naval Underwater Systems Center, New London, CT, 25 October 1977.

[11] I. Kanter, "Exact Detection Probability for Partially Correlated Rayleigh Targets", **IEEE Transactions on Aerospace and Electronic Systems**, volume AES-22, number 2, pages 184 - 195, March 1986.

[12] A. H. Nuttall and E. S. Eby, **Signal-to-Noise Ratio Requirements for Detection of Multiple Pulses Subject to Partially Correlated Fading With Chi-Squared Statistics of Various Degrees of Freedom**, NUSC Technical Report 7707, Naval Underwater Systems Center, New London, CT, 2 June 1986.

[13] S. Watts, "Radar Detection Prediction in K-Distributed Sea Clutter and Thermal Noise", **IEEE Transactions on Aerospace and Electronic Systems**, volume AES-23, number 1, pages 40 - 45, January 1987.

[14] A. H. Nuttall, **Detection Performance of Normalizer for Multiple Signals Subject to Partially Correlated Fading With Chi-Squared Statistics**, NUSC Technical Report 8133, Naval Underwater Systems Center, New London, CT, 1 October 1987.

[15] X. Y. Hou and N. Morinaga, "Detection Performance in K-Distributed and Correlated Rayleigh Clutter", **IEEE Transactions on Aerospace and Electronic Systems**, volume AES-25, number 5, pages 634 - 642, September 1989.

[16] I. S. Gradshteyn and I. M. Ryzhik, **Table of Integrals, Series, and Products**, Academic Press, Inc., New York, NY, 1980.

[17] A. H. Nuttall, **Evaluation of Densities and Distributions via Hermite and Generalized Laguerre Series Employing High-Order Expansion Coefficients Determined Recursively via Moments or Cumulants**, NUSC Technical Report 7377, Naval Underwater Systems Center, New London, CT, 28 February 1985.

[18] F. B. Hildebrand, **Methods of Applied Mathematics**, Prentice-Hall, Inc., New York, NY, 1954.

[19] C. W. Helstrom, **Statistical Theory of Signal Detection**, second edition, Pergamon Press, Inc., New York, NY, 1968.

[20] W. B. Davenport and W. L. Root, **Random Signals and Noise**, McGraw-Hill Book Company, Inc., New York, NY, 1958.

[21] A. H. Nuttall, **Accurate Efficient Evaluation of Cumulative or Exceedance Probability Distributions Directly From Characteristic Functions**, NUSC Technical Report 7023, Naval Underwater Systems Center, New London, CT, 1 October 1983.

[22] G. A. Korn and T. M. Korn, **Mathematical Handbook for Scientists and Engineers**, McGraw-Hill Book Co., Inc., New York, NY, 1961.

INITIAL DISTRIBUTION LIST

Addressee	No. of Copies
Admiralty Underwater Weapons Establishment, England	
Library	1
Center for Naval Analyses	1
Coast Guard Academy	
Prof. J. J. Wolcin	1
Defense Nuclear Agency, RAAE	
J. Meyers	1
Defence Research Establishment Atlantic, Nova Scotia	
B. E. Mackey (Library)	1
Dr. S. Stergiopoulos	1
Defence Research Establishment Pacific, British Columbia	
Dr. D. J. Thomson	1
Defence Science Establishment, HMNZ Dockyard, New Zealand	
Director	1
Defence Scientific Establishment, New Zealand	
Dr. L. H. Hall	1
Defense Advanced Research Projects Agency	
Commanding Officer	1
A. W. Ellinthorpe	1
Defense Technical Information Center	12
Dept. of Science & Industrial Research, New Zealand	
M. A. Poletti	1
National Radio Astronomy Observatory	
F. Schwab	1
National Security Agency	
Dr. J. R. Maar (R51)	1
Naval Air Development Center	
Commander	1
A. Witt	1
T. Madera	1
L. Allen (Code 50)	1
Naval Air Systems Command	
NAIR-93	1
Naval Command Control and Ocean Surveillance Center, Hawaii	1
Naval Command Control and Ocean Surveillance Center, San Diego	
Commanding Officer	1
C. Tran	1
J. M. Alsup (Code 635)	1
J. Silva	1
D. Hanna	1
P. Nachtigall	1
W. Marsh	1
C. Persons	1
Naval Environmental Prediction Research Facility	1
Naval Intelligence Command	1
Naval Oceanographic Office	1
Naval Oceanographic & Atmospheric Research Laboratory, CA	
M. J. Pastore	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Naval Oceanographic & Atmospheric Research Laboratory, MS	
Commanding Officer	1
R. Wagstaff (Code 245)	1
B. Adams	1
E. Franchi	1
R. Fiddler (Code 245)	1
Naval Personnel Research & Development Center	1
Naval Postgraduate School	
Superintendent	2
Prof. C. W. Therrien (Code 62 TI)	1
Naval Research Laboratory, Orlando, USRD	1
Naval Research Laboratory, Washington	
Commanding Officer	1
Dr. P. B. Abraham (Code 5131)	1
W. F. Gabriel (Code 5370)	1
A. A. Gerlach	1
N. Yen (Code 5130)	1
D. Bradley	1
S. Sachs	1
D. Steiger	1
E. Wald (Code 5150)	1
Naval Sea Systems Command	
SEA-00; -63; -63D; -63X; -92R; PMS-402	6
Naval Surface Warfare Center, Dahlgren, VA	
J. Gray (Code G71)	1
Naval Surface Warfare Center, Bethesda, MD	
Commander	1
H. Chaplin	1
L. Becker	1
W. Phillips (Code 1932)	1
Naval Surface Warfare Center, Annapolis, MD	
P. Prendergast (Code 2744)	1
Naval Surface Warfare Center, Coastal Systems Station	
Commanding Officer	1
D. Skinner	1
E. Linsenmeyer	1
Naval Surface Weapons Center, Dahlgren, VA	
Commander	1
H. Crisp	1
D. Phillips	1
T. Ryczek	1
Naval Surface Weapons Center, White Oak Lab.	1
M. Strippling	1
Naval Surface Weapons Center, Fort Lauderdale	1
Naval Technical Intelligence Center	
Commanding Officer	2
D. Rothenberger	1
Naval Undersea Warfare Center Detachment, West Palm Beach	
Officer-in-Charge	1
Dr. R. M. Kennedy (Code 3802)	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Naval Weapons Center	1
Norwegian Defence Research Establishment	
Dr. J. Glattetre	1
Office of the Chief of Naval Research, Arlington, VA	
OCNR-00; -10; -11; -12; -13; -20; -21; -22; -23 (3)	11
N. L. Gerr (Code 1111)	1
A. Wood	1
D. Johnson	1
SACLANT Undersea Research Center	
Prof. G. Tacconi	1
Library	1
Sonar and Surveillance Group, RANRL, Australia	1
Space & Naval Warfare System Command	
SPAWAR-00; -04; -005; PD-80; PMW-181	5
L. Parrish	1
R. Cockerill	1
U.S. Air Force, Alabama	
Air University Library	1
U.S. Coast Guard Research & Development Center	
Library	1
U.S. Department of Commerce, NTIA/ITS	
Dr. A. D. Spaulding	1
Weapons Systems Research Laboratory, Australia	
HASC	1
HSPC	1
Brown University	
Documents Library	1
Canberra College of Advanced Education	
P. Morgan	1
Concordia University, Quebec	
Prof. J. Krolik	1
Dalhousie University	
Dr. B. Ruddick	1
Drexel University	
Prof. S. B. Kesler	1
Prof. P. R. Chitrapu	1
Harvard University	
Gordon McKay Library	1
Indian Institute of Technology, India	
Dr. K. M. M. Prabhu	1
Johns Hopkins University, Applied Physics Laboratory	
Director	1
J. C. Stapleton	1
Kansas State University	
Prof. B. Harms	1
Lawrence Livermore National Laboratory	
Director	1
L. C. Ng	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Los Alamos National Laboratory	1
Marine Biological Laboratory, Woods Hole	1
Marine Physical Laboratory, Scripps	1
Massachusetts Institute of Technology	
Prof. A. B. Baggaroer	1
Barker Engineering Library	1
Northeastern University	
Prof. C. L. Nikias	1
Penn State University, Applied Research Laboratory	
Director	1
F. W. Symons	1
R. Hettche	1
E. Liszka	1
Purdue University	
Prof. N. Srinivasa	1
Queensland University of Technology	
Prof. B. Boashash	1
Royal Military College of Canada	
Prof. Y. T. Chan	1
Rutgers University	
Prof. S. Orfanidis	1
San Diego State University	
Prof. F. J. Harris	1
Sandia National Laboratory	
Director	1
J. Claasen (315)	1
Simon Fraser University	
Dr. E. F. Velez	1
Southeastern Massachusetts University	
Prof. C. H. Chen	1
State University of New York	
Prof. M. Barkat	1
Syracuse University	
Prof. D. Weiner	1
Tel-Aviv University, Israel	
Prof. E. Weinstein	1
United Engineering Center	
Engineering Societies Library	1
University of Alberta, Canada	
K. L. Yeung	1
University of Auckland	
Dr. M. D. Johns	1
University of California, San Diego	
Prof. C. W. Helstrom	1
University of Colorado	
Prof. L. L. Scharf	1
University of Connecticut	
Prof. C. H. Knapp	1
Wilbur Cross Library	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
University of Florida	
Prof. D. C. Childers	1
University of Illinois	
Dr. D. L. Jones	1
University of Michigan	
Communications & Signal Processing Laboratory	1
W. J. Williams	1
University of Minnesota	
Prof. M. Kaveh	1
University of Rhode Island	
Prof. G. F. Boudreaux-Bartels	1
Prof. S. M. Kay	1
Prof. D. Tufts	1
Library	1
University of Rochester	
Prof. E. Titlebaum	1
University of Southern California	
Prof. W. C. Lindsey	1
Dr. A. Polydoros	1
University of Strathclyde, Scotland	
Prof. T. S. Durrani	1
University of Technology, England	
Prof. J. W. R. Griffiths	1
University of Texas, Applied Research Laboratory	1
University of Washington	
Applied Physics Laboratory	1
Prof. D. W. Lytle	1
Dr. R. C. Spindel	1
Villanova University	
Prof. M. G. Amin	1
Woods Hole Oceanographic Institution	
Director	1
Dr. E. Weinstein	1
Yale University	
Prof. A. Nehorai	1
Prof. P. M. Schultheiss	1
Prof. F. Tuteur	1
Kline Science Library	1
A&T, North Stonington, CT	
H. Jarvis	1
A&T, Arlington, VA	
D. L. Clark	1
Accurate Automation	
P. K. Simpson	1
BB&N, Cambridge, MA	
H. Gish	1
BB&N, New London, CT	
Dr. P. G. Cable	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Bell Communications Research	
J. F. Kaiser	1
D. Sunday	1
Berkeley Research Associates	
Steven W. McDonald	1
Cogent Systems, Inc.	
J. P. Costas	1
EDO Corporation	
M. Blanchard	1
EG&G	
D. Frohman	1
General Electric Company, Moorestown, NJ	
Dr. M. R. Allen	1
H. Urkowitz	1
General Electric Company, Philadelphia, PA	
T. J. McFall	1
General Electric Company, Pittsfield, MA	
R. W. Race	1
General Electric Company, Syracuse, NY	
Dr. A. M. Vural	1
D. Winfield	1
Harris Scientific Services	
B. Harris	1
Hughes Aircraft Company, Buena Park, CA	
T. E. Posch	1
IBM	
G. L. Demuth	1
Interstate Electronics Corporation	
R. O. Nielsen (Dept. 8011)	1
Kildare Corporation	
Dr. R. Mellen	1
Lincom Corporation	
Dr. T. A. Schonhoff	1
Magnavox Elec. Systems Company	
R. Kenefic (Dept. 525)	1
MSB Systems, Inc.	
A. Winder	1
Nichols Research Corporation	
Dr. T. L. Marzetta	1
Orincon Corporation, Columbia, MD	
S. L. Marple	1
Orincon Corporation, San Diego, CA	
Dr. J. W. Young	1
Orincon Corporation, Arlington, VA	
Dr. H. Cox	1
Prometheus, Inc., Newport, RI	
M. J. Barrett	1
Prometheus, Inc., Sharon, MA	
Dr. J. S. Byrnes	1
RAN Research Lab, Australia	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Raytheon, Portsmouth, RI	
J. Bartram	1
R. Conner	1
S. S. Reese	1
Rockwell International	
L. T. Einstein	1
Dr. D. F. Elliott	1
SAIC, McLean, VA	
Dr. P. Mikhalevsky	1
SAIC, New London	
Dr. F. DiNapoli	1
SIMRAD SUBSEA A/S, Naval Systems Div.	
E. B. Lunde	1
Sperry Corporation, Defense Marine Systems Unit	1
Toyon Research Corporation	
M. L. Van Blaricum	1
Tracor, Austin, TX	
Dr. T. J. Leih	1
B. Jones	1
Dr. K. Scarbrough	1
TRW, Fairfax, VA	
R. Prager	1
G. C. Maher	1
Welse Sistemi Subacquei, Genova, Italy	
Dr. H. Van Asselt	1
Westinghouse Electric Corporation, Annapolis, MD	
H. Newman	1
Dr. H. L. Price	1
Westinghouse Electric Corporation, Baltimore, MD	
R. Park	1
Westinghouse Electric Corporation, Waltham, MA	
R. B. Kennedy	1
Assard, G. L.	1
Beidas, B.	1
Bendat, Dr. J. S.	1
Bleistein, Dr. N.	1
Breton, R.	1
Cohen, Dr. L.	1
Hahn, W. R.	1
Lloyd, L.	1
Maltz, F.	1
Middleton, Dr. D.	1
Nash, H. E.	1
Nicholson, D.	1
O'Brien, W.	1
Papoutsanis, P.D.	1
Pohler, R. F.	1
Price, Dr. R.	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Raisbeck, Dr. G.	1
Richter, W.	1
Schulkin, Dr. M.	1
Urick, R. J.	1
Von Winkle, Dr. W. A.	1
Werbner, A.	1
Wilson, Dr. J. H.	1